Differential approximation results for the traveling salesman problem with distances 1 and 2

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Abstract

We prove that both minimum and maximum traveling salesman problems on complete graphs with edge-distances 1 and 2 (denoted by min_TSP12 and max_TSP12, respectively) are approximable within 3/4. Based upon this result, we improve the standard approximation ratio known for maximum traveling salesman with distances 1 and 2 from 3/4 to 7/8. Finally, we prove that, for any $\epsilon > 0$, it is **NP**-hard to approximate both problems better than within $741/742 + \epsilon$. The same results hold when dealing with a generalization of min_ and max_TSP12, where instead of 1 and 2, edges are valued by a and b.

1 Introduction

Given a complete graph on n vertices, denoted by K_n, with edge distances either 1 or 2 the minimum traveling salesman problem (min_TSP12) consists of minimizing the cost of a Hamiltonian cycle, the cost of such a cycle being the sum of the distances on its edges (in other words, in finding a Hamiltonian cycle containing a maximum number of 1-edges). The maximum traveling salesman problem (max_TSP) consists of maximizing the cost of a Hamiltonian cycle (in other words, of finding a Hamiltonian cycle containing a maximum number of 2-edges). A generalization of TSP12, denoted by TSPab, is the one where the edge-distances are either a, or b, $a < b$. Both min_ and max_TSP12, and TSPab are **NP**-hard.

Given an instance I of an **NP** optimization (**NPO**) problem Π and a polynomial time approximation algorithm $\mathtt{A}$ feasibly solving Π, we will denote by $\omega(I)$, $\lambda_{\mathtt{A}}(I)$ and $\beta(I)$ the values of the worst solution of I, of the approximated one (provided by $\mathtt{A}$ when running on I), and the optimal one for I, respectively. Generally (see [8]), the quality of an approximation algorithm for an **NP**-hard minimization (resp., maximization) problem Π is expressed by the ratio (called standard in what follows) $\rho_{\mathtt{A}}(I) = \lambda(I)/\beta(I)$, and the quantity $\rho_{\mathtt{A}} = \inf\{r : \rho_{\mathtt{A}}(I) < r, I \text{ instance of } \Pi\}$ (resp., $\rho_{\mathtt{A}} = \sup\{r : \rho_{\mathtt{A}}(I) > r, I \text{ instance of } \Pi\}$) constitutes the approximation ratio of $\mathtt{A}$ for Π. Another approximation-quality criterion used by many researchers ([2, 1, 3, 4, 13, 14]) is what in [6, 5] we call *differential-approximation ratio*. It measures how the value of an approximate solution is placed in the interval between $\omega(I)$ and $\beta(I)$. More formally, the differential-approximation ratio of an algorithm $\mathtt{A}$ is defined as $\delta_{\mathtt{A}}(I) = |\omega(I) - \lambda(I)|/|\omega(I) - \beta(I)|$. The quantity $\delta_{\mathtt{A}} = \sup\{r : \delta_{\mathtt{A}}(I) > r, I \text{ instance of } \Pi\}$ is the differential approximation ratio of $\mathtt{A}$ for Π. In [2], the term "trivial solution" is used to denote the solution realizing the worst among the feasible solution-values of an instance. Moreover, all the examples in [2] carry over **NP**-hard problems for which worst solution can be trivially computed. This is for example the case of maximum independent set where, given a graph, the worst solution is the empty set, or of minimum vertex cover, where the worst solution is the vertex-set of the input-graph, or even of the

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minimum graph-coloring where one can trivially color the vertices of the input-graph using a distinct color per vertex. On the contrary, for TSP things are very different. Let us take for example min_TSP. Here, given a graph K_n, the worst solution for K_n is a maximum total-distance Hamiltonian cycle, i.e., the optimal solution of max_TSP in K_n. The computation of such a solution is very far from being trivial since max_TSP is **NP**-hard. Obviously, the same holds when one considers max_TSP and tries to compute a worst solution for its instance, as well as for optimum satisfiability, for minimum maximal independent set and for many other well-known **NP**-hard problems. In order to remove ambiguities about the concept of the worst-value solution of an instance I of an **NPO** problem Π, we will define it as the optimal solution $\text{opt}(\Pi')$ of an **NPO** problem Π' having the same set of instances and feasibility constraints as Π verifying

$$\text{opt}(\Pi') = \begin{cases} \max & \text{opt}(\Pi) = \min \\ \min & \text{opt}(\Pi) = \max \end{cases}$$

In general, no apparent links exist between standard and differential approximations in the case of minimization problems, in the sense that there is no evident transfer of a positive, or negative, result from one framework to the other. Hence, a "good" differential-approximation result implies nothing for the behavior of the approximation algorithm studied when dealing with the standard framework and vice versa. When dealing with maximization problems, we show in [10] that the approximation of a maximization **NPO** problem Π within differential-approximation ratio δ implies its approximation within standard-approximation ratio δ.

The best known standard-approximation ratio known for min_TSP12 is 7/6 ([11]), while the best known standard inapproximability bound is $743/742 - \epsilon$, for any $\epsilon > 0$ ([7]). On the other hand, the best known standard-ratio max_TSP is 3/4 ([12]). To our knowledge, no better result is known in standard approximation for max_TSP12. Furthermore, no special study of TSPab has been performed until now (a trivial standard-approximation ratio of b/a or a/b is in any case very easily deduced for min_ or max_TSPab).

Here we show that min_ and max_TSP12, and min_ and max_TSPab are all equi-approximable within 3/4 for the differential approximation. We also prove that all these problems cannot be approximated better than within $741/742 + \epsilon$, for any $\epsilon > 0$. By the equi-approximability of min_TSP12, max_TSP12, min_TSPab and max_TSPab, the results obtained for the case of min_TSP12 apply to the rest of the problems above. Finally, we improve the *standard*-approximation ratio of max_TSP12 from 3/4 ([12]) to 7/8.

In what follows, we will denote by $V = \{v_1, \ldots, v_n\}$ the vertex-set of K_n, by E its edge-set, and for $v_iv_j \in E$, we denote by $d(v_i, v_j)$ the distance of the edge $v_iv_j \in E$; we consider that the distance-vector is symmetric and integer. Given a feasible TSP-solution $T(K_n)$ of K_n (both min_ and max_TSP have the same set of feasible solutions), we denote by $d(T(K_n))$ its (objective) value. Given a graph G, we denote by $V(G)$ its vertex-set. Finally, given any set C of edges, we denote by $d(C)$ the total distance of C, i.e., the quantity $\sum_{v_iv_j \in C} d(v_i, v_j)$.

2 Differential-approximation preserving reductions for TSP12

In this section we give a differential-approximation preserving result that will be used later.

Theorem 1. *min_TSP12, max_TSP12, min_TSPab and max_TSPab are all equi-approximable for the differential approximation.*

Proof. In order to prove the theorem we will prove the following stronger lemma.

Lemma 1. *Consider any instance $I = (K_n, \vec{d})$ (where $\vec{d}$ denotes the edge-distance vector of K_n). Then, any legal transformation $\vec{d} \mapsto \gamma.\vec{d} + \eta.\vec{1}$ of $\vec{d}$ ($\gamma, \eta \in \mathbb{Q}$) produces differentially equi-approximable TSP-problems.*

Proof of lemma 1. Suppose that TSP can be approximately solved within differential-approximation ratio δ and remark that both the initial and the transformed instance have the same set of feasible solutions. By the transformation considered, the value $d(T(K_n))$ of any feasible tour $T(K_n)$ is affinely transformed into $\gamma d(T(K_n)) + \eta n$. Since differential-approximation ratio is stable under affine transformation, the equi-approximability of the original and of the transformed problem is immediately deduced, concluding so the proof of lemma 1. ∎

We are ready now to continue the proof of theorem 1. In order to prove that min_TSP12 and max_TSP12 are equi-approximable, it suffices to apply lemma 1 proved just above with $\gamma = -1$ and $\eta = 3$. On the other hand, in order to prove that min_ or max_TSP12 reduces to min_ or max_TSPab, we apply lemma 1 with $\gamma = 1/(b-a)$ and $\eta = (b-2a)/(b-a)$, while for the converse reduction we apply lemma 1 with $\gamma = b - a$ and $\eta = 2a - b$. Since the reductions presented are transitive and composable, the equi-approximability of the pairs (min_TSP12, max_TSP12) and (TSP12, TSPab) proves the theorem. ∎

For reasons of simplicity, we deal, in what follows, with min_TSP12. The differential-approximation results obtained can be immediately transferred, by theorem 1, to max_TSP12, min_TSPab and max_TSPab.

3 Approximating min_TSP12

Let us first recall that, given a graph G, a 2-matching is a set M of edges of G such that if $V(M)$ is the set of the endpoints of M, the vertices of the graph $(V(M), M)$ have degree at most 2; in other words, the graph $(V(M), M)$ is a collection of cycles and simple paths. A 2-matching is optimal if it is the largest over all the 2-matchings of G. It is called perfect if any vertex of the graph $(V(M), M)$ has degree equal to 2, i.e., if it constitutes a partition of $V(M)$ into cycles in G. Remark that determining a maximum 2-matching in a graph G is equivalent to determining a minimum total-distance vertex-partition into cycles into $G \cup \bar{G}$ (the complement of G), where the edges of G are considered of distance 1 and the ones of $\bar{G}$ of distance 2.

As shown in [9], *an optimal triangle-free 2-matching can be computed in polynomial time.* As mentioned above, this amounts to computing a triangle-free minimum-distance collection of cycles in a complete graph K_n with edge-distances 1 and 2. Let us denote by M such a collection. Starting from M, we will progressively patch its cycles in order to finally obtain a unique Hamiltonian cycle in K_n.

3.1 Preprocessing M

We first define two operations, namely the *2-exchange* and the *2-patching*, implying two vertex-disjoint cycles of a 2-matching.

Definition 1. Let C_1 and C_2 be two vertex-disjoint cycles. Then:

- a *2-exchange* is any replacement of two edges $v_1u_1 \in C_1$, $v_2u_2 \in C_2$ by the edges v_1v_2 and u_1u_2;
- a *2-patching* of C_1 and C_2 is any cycle C resulting from a 2-exchange on C_1 and C_2, i.e., $C = (C_1 \cup C_2 \setminus \{v_1u_1, v_2u_2\}) \cup \{v_1v_2, u_1u_2\}$, for any pair $(v_1u_1, v_2u_2) \in C_1 \times C_2$. ∎

A 2-matching minimal with respect to the 2-exchange operation will be called *2-minimal*. In particular, if all edges have the same cost, then a 2-minimal matching is a tour.

Definition 2. A 2-matching $M = (C_1, C_2, \ldots, C_{|M|})$ is 2-minimal if it verifies, $\forall (C_i, C_j) \in M \times M$, $C_i \neq C_j$, $\forall v_1u_1 \in C_i$, $\forall v_2u_2 \in C_j$, $d(v_1, v_2) + d(u_1, u_2) > d(u_1v_1) + d(u_2v_2)$. ∎

In other words, a 2-matching M is 2-minimal if any 2-patching of its cycles produces a 2-matching of total distance strictly greater than the one of M. Starting from a 2-matching $\hat{M}$ transformation of $\hat{M}$ into a 2-minimal one M can be performed in polynomial time by the following procedure.

```
BEGIN *2_MIN*
      M_p ← ∅;
      REPEAT
             pick a new set {C_i, C_j} ⊆ M̂;
             FOR all v_k v_l ∈ C_i, v_p v_q ∈ C_j DO
                 take edges v_k v_l ∈ C_i and v_p v_q ∈ C_j;
                 C^1_ij ← C_i ∪ C_j \ {v_k v_l, v_p v_q} ∪ {v_k v_p, v_l v_q};
                 C^2_ij ← C_i ∪ C_j \ {v_k v_l, v_p v_q} ∪ {v_k v_q, v_l v_p};
                 C_ij ← argmin{d(C^1_ij), d(C^2_ij)};
                 M_p ← M̂ \ {C_i, C_j} ∪ C_ij;
                 IF d(M̂) > d(M_p) THEN M̂ ← M_p FI
             OD
      UNTIL no improvement of d(M̂) is possible;
      OUTPUT M ← M̂;
END. *2_MIN*
```

Moreover, suppose that there exist two distinct cycles C and C' of M (the output of the procedure 2_MIN), both containing 2-edges and denote by $uv \in C$ and $u'v' \in C'$ two such edges. Then, $d(uu') + d(vv') \geqslant 4$, while $d(uv) + d(u'v') = 4$, a contradiction. So, the following proposition holds.

Proposition 1. *In any 2-minimal 2-matching, at most one of its cycles contains 2-edges.*

Remark 1. If the size of a 2-minimal triangle-free 2-matching M is 1, then, since a Hamiltonian tour is a special case of triangle-free 2-matching, M is an optimal min_TSP12-solution. Hence, in what follows we will suppose 2-matchings of size at least 2. ∎

Assume now a 2-minimal triangle-free 2-matching $M = (C_1, \dots, C_p, C_0)$, verifying remark 1, where C_0 is the unique cycle of M (if any) containing 2-edges. Construct a graph $H = (V_H, E_H)$; $V_H = \{w_1, \dots, w_p\}$ contains a vertex per cycle of M and, for $i \neq j$, $w_i w_j \in E_H$ iff $\exists (u, v) \in C_i \times C_j$ such that $d(u, v) = 1$. Consider a maximum matching M_H, $|M_H| = q$, of H. With any edge $w_{i^s} w_{j^s}$ of M_H we associate the pair (C_{i^s}, C_{j^s}) of the corresponding cycles of M. So, M can be described (up to renaming its cycles) as

$$M = \bigcup_{s=1}^{q} \{C_1^s, C_2^s\} \bigcup_{t=1}^{r=p-2q} \{C_t\} \bigcup \{C_0\} \tag{1}$$

where for $s = 1, \dots, q$, $\exists e^s \in V(C_1^s) \times V(C_2^s)$ such that $d(e^s) = 1$.

Consider M as expressed in (1), denote by V_s the set of the four vertices of C_1^s and C_2^s adjacent to the endpoints of e^s, and construct the bipartite graph $B = (V_B^1 \cup V_B^2, E_B)$ where $V_B^1 = \{w_1, \dots, w_r\}$ (i.e., we associate a vertex with a cycle C_t, $t = 1, \dots, r$), $V_B^2 = \{w^1, \dots, w^q\}$ (i.e., we associate a vertex with a pair (C_1^s, C_2^s), $s = 1, \dots q$) and, $\forall (t, s)$, $w_t w^s \in E_B$ iff $\exists u \in C_t$, $\exists v \in V_s$ such that $d(u, v) = 1$. Compute a maximum matching M_B, $|M_B| = q'$ in B. With any edge $w_t w^s \in M_B$ we associate the triple (C_1^s, C_2^s, C_t). So, M can be described (up to renaming its cycles) as

$$M = \bigcup_{s=1}^{q'} \{C_1^s, C_2^s, C_3^s\} \bigcup_{s=q'+1}^{q} \{C_1^s, C_2^s\} \bigcup_{t=1}^{r'=r-q'} \{C_t\} \bigcup \{C_0\} \tag{2}$$

where for $s = 1, \ldots, q'$, $\exists f^s \in V_s \times V(C_3^s)$ such that $d(f^s) = 1$. In what follows we will reason with respect to M as it has been expressed in (2).

3.2 Computation and evaluation of the approximate solution and a lower bound for the optimal tour

In the sequel, call s.d.e.p. a set of vertex-disjoint elementary paths, denote by PREPROCESS the procedure that starting from a 2-minimal triangle-free 2-matching M leads to (2) and consider the following algorithm.

```
BEGIN (*TSP12*)
      compute a maximum 2-matching M̂ in K_n;
      M ← 2_MIN(M̂);
      M ← PREPROCESS(M);
      D ← ∅;
(1)   FOR s ← 1 TO q' DO
          let g_1^s be the edge of C_1^s adjacent to both e^s and f^s;
          choose in C_2^s an edge g_2^s adjacent to e^s;
          choose in C_3^s an edge g_3^s adjacent to f^s;
          D ← (D ∪ C_1^s ∪ C_2^s ∪ C_3^s \ {g_1^s, g_2^s, g_3^s}) ∪ {e^s, f^s};
          OD
(2)   FOR s ← q' + 1 TO q DO
          choose in C_1^s an edge g_1^s adjacent to e^s;
          choose in C_2^s an edge g_2^s adjacent to e^s;
          D ← (D ∪ C_1^s ∪ C_2^s \ {g_1^s, g_2^s}) ∪ {e^s};
          OD
(3)   FOR t ← 1 TO r' DO
          choose any edge g_t in C_t;
          D ← (D ∪ C_t) \ {g_t};
          OD
(4)   IF there exists in C_0 an 1-edge
        THEN choose a 2-edge g_0 of C_0 adjacent to an 1-edge e;
        ELSE choose any edge g_0 of C_0;
        D ← (D ∪ C_0) \ {g_0};
      FI
(5)   complete D in order to obtain a Hamiltonian tour T(K_n);
      OUTPUT T(K_n);
END (*TSP12*)
```

Clearly, both achievement of a 2-minimal triangle free 2-matching and PREPROCESS can be performed in polynomial time. Moreover, steps (1) to (4) are also executed in polynomial time. Finally, step (5) can be performed by arbitrarily ordering (mod$|D|$) the chains of the s.d.e.p. D and then, for $i = 1, \ldots, |D|$, adding in D the edge linking the "last" vertex of chain i to the "first" vertex of chain $i + 1$. Consequently, the whole algorithm TSP12 is polynomial. Finally, remark that C_0 may contain either only 2-edges, or both 1- and 2-edges. In the latter case, edge g_0 (in step 4) can be any 2-edge in C_0, adjacent to an 1-edge of C_0.

Lemma 2. $d(T(K_n)) \leqslant d(M) + q + r'$.

Proof. During steps (1) to (4) of algorithm TSP12, set D remains a s.d.e.p. At the end of step (4), D contains M minus the $3q' + 2(q - q') + r' = q' + 2q + r'$ 1-edges of the set

$\cup_{s=1}^{q'}\{g_1^s, g_2^s, g_3^s\} \cup_{s=q'+1}^{q}\{g_1^s, g_2^s\} \cup_{s=1}^{r'}\{g_t\}$ minus (if $C_0 \neq \emptyset$) one 2-edge of C_0 plus the $2q'+(q-q') = q'+q$ 1-edges of the set $\cup_{s=1}^{q'}\{e_1^s, f_2^s\} \cup_{s=q'+1}^{q}\{e^s\}$. So D is a s.d.e.p. of size $n-(q+r')-\mathbf{1}_{C_0\neq\emptyset}$ and of total distance $d(M)-(q+r')-2.\mathbf{1}_{C_0\neq\emptyset}$. Completion of D in order to obtain a tour in K_n, can be done by adding $q+r'+\mathbf{1}_{C_0\neq\emptyset}$ new edges. Each of these new edges can be, at worst, of distance 2. We so have $d(T(K_n)) \leqslant d(M)-(q+r'+2.\mathbf{1}_{C_0\neq\emptyset})+2(q+r'+\mathbf{1}_{C_0\neq\emptyset}) = d(M)+q+r'$, q.e.d. ∎

On the other hand, the optimal tour being a special triangle-free 2-matching, the following lemma holds immediately.

Lemma 3. *$\beta(K_n) \geqslant d(M)$.*

3.3 Evaluation of the worst-value solution

In what follows in this section we will exhibit a s.d.e.p., with all edges of distance 2 (called 2-s.d.e.p.). Given such a s.d.e.p. W, one can proceed as in step `(5)` of algorithm `TSP12` (section 3.2), in order to construct a Hamiltonian tour T_w whose total distance is a lower bound for $\omega(K_n)$.

Denote by $E2$ the set of 2-edges of cycle C_0. If $q=0$, i.e., $M_H=\emptyset$, and if $C_0=E2$, then the tour computed by `TSP12` is optimal.

Lemma 4. *If $q=0$ and $C_0=E2$, then $\delta_{\texttt{TSP12}}(K_n)=1$.*

Proof. Let $k=|V(C_0)|=d(M)-n$ and set $V(C_0)=\{a_1,\ldots,a_k\}$. By the fact that M is 2-minimal, all the edges of K_n incident to these vertices have distance 2. On the other hand, between two distinct cycles in the set $\{C_1,\ldots,C_{p=r'}\}$ of M, there exist only edges of distance 2. Consider the family $\mathcal{F}=\{\{a_1\},\ldots,\{a_k\},V(C_1),\ldots V(C_p)\}$. By the above remarks, any edge linking vertices of two distinct sets of $\mathcal{F}$ is a 2-edge. Any feasible tour of K_n (a posteriori an optimal one) integrates the $k+p$ sets of $\mathcal{F}$ by using at least $k+p$ 2-edges pairwise linking these sets. Hence, any tour uses at least $k+p$ 2-edges, so does tour $T(K_n)$ computed by algorithm `TSP12`, q.e.d. ∎

So, we suppose in the sequel that $q=0 \Rightarrow C_0 \neq E_2$. We will now prove the existence of a 2-s.d.e.p. W of size $d(M)+4(q+r')-n$, where M is as expressed by (2). In the sequel, a path with alternating vertices will denote a path such that no two consecutive vertices lie in the same cycle.

Proposition 2. *Between two cycles C_a and C_b of M of size at least k, there always exists a path with alternating vertices from C_a and C_b, which contains at least k 2-edges.*

Proof. Let $\{a_1,\ldots,a_{k+1}\}$ and $\{b_1,\ldots,b_{k+1}\}$ be $k+1$ successive vertices of two distinct cycles C_a and C_b of size at least k (possibly $a_1=a_{k+1}$ if $|V(C_a)|=k$ and $b_1=b_{k+1}$ if $|V(C_b)|=k$). We will show that there exists a path with alternating vertices from C_a and C_b of size $2k-1$ and of distance at least $3k-1$. Consider paths $C=\cup_{i=1}^{k}\{a_ib_i\}\cup_{i=1}^{k-1}\{a_{i+1}b_i\}$ and $D=\cup_{i=2}^{k+1}\{a_ib_i\}\cup_{i=1}^{k-1}\{a_ib_{i+1}\}$. By the 2-minimality of M we get:

$$\begin{array}{lll} \forall i=1,\ldots,k & \max\{d(a_i,b_i),d(a_{i+1},b_{i+1})\}=2 & \Rightarrow\ d(a_i,b_i)+d(a_{i+1},b_{i+1})\geqslant 3 \\ \forall i=1,\ldots,k-1 & \max\{d(a_i,b_{i+1}),d(a_{i+1},b_i)\}=2 & \Rightarrow\ d(a_i,b_{i+1})+d(a_{i+1},b_i)\geqslant 3 \end{array}$$

Summing the terms of the expression above member-by-member, one obtains:

$$\sum_{i=1}^{k}\left(d(a_i,b_i)+d(a_{i+1},b_{i+1})\right)+\sum_{i=1}^{k-1}\left(d(a_{i+1},b_i)+d(a_i,b_{i+1})\right)\geqslant 6k-3$$

$$\iff\ d(C)+d(D)\geqslant 6k-3\ \Rightarrow\ \max\{d(C),d(D)\}\geqslant\left\lceil\frac{6k-3}{2}\right\rceil=3k-1 \quad ∎$$

Application of proposition 2 to any pair (C_1^s, C_2^s) of M results in the following claim.

Claim 1. *$\forall s = 1, \dots, q$, there exists a 2-s.d.e.p. W^s of size 4, alternating vertices of cycles C_1^s and C_2^s, containing a vertex of V_s whose degree with respect to W^s is 1.*

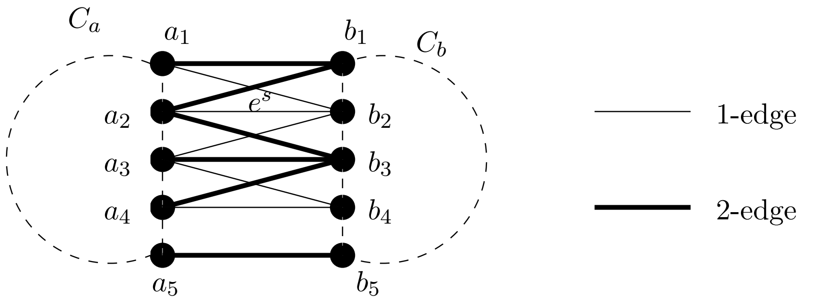

Figure 1: An example of claim 1.

In figure 1, we show an application of claim 1. We assume $e^s = a_2b_2$; then $\{a_1, b_1\} \subset V_s$. The 2-s.d.e.p. W^s claimed is $\{b_1a_2, (a_3b_3, b_3a_4), a_5b_5\}$ and the degree of b_1 with respect to W^s is 1.

Consider now the s.d.e.p. $W_t^s = W^s \cup W'^s_t$, where W^s as in claim 1 and W'^s_t is any path of size 4 with alternating vertices from C_3^s and C_t, $s = 1, \dots, q'$, $t = 1, \dots, r'$. By the optimality of M_H, any edge linking vertices of C_3^s to vertices of C_t is a 2-edge. Consequently, W_t^s is a 2-s.d.e.p. and the following claim holds.

Claim 2. *$\forall s = 1, \dots, q'$, $\forall t = 1, \dots, r'$, there exists a 2-s.d.e.p. W_t^s of size 8, alternating vertices of the cycles C_1^s and C_2^s, and of the cycles C_3^s and C_t.*

For $s = q' + 1, \dots, q$, $t = 1, \dots r'$, consider the triplet (C_1^s, C_2^s, C_t). Let $e^s = e_1^s e_2^s$, $V_s = \{u_1^s, v_1^s, u_2^s, v_2^s\}$ and consider any four vertices a_t, b_t, c_t and d_t of C_t. By the optimality of M_B, any vertex of C_t is linked to any vertex of V^s exclusively by 2-edges. Moreover, the 2-minimality of M implies that at least one of $u_1^s e_2^s$ and $e_1^s u_2^s$ is of distance 2. If we suppose $d(u_1^s, e_2^s) = 2$ (figure 2), then the path $\{e_2^s, u_1^s, a_t, v_1^s, b_t, u_2^s, c_t, v_2^s, d_t\}$ is a 2-s.e.d.p. Hence, the following claim holds.

Claim 3. *$\forall s = q' + 1, \dots, q$, $\forall t = 1, \dots, r'$, there exists a 2-s.d.e.p. W_t^s of size 8, alternating vertices of the cycles C_1^s, C_2^s and C_t.*

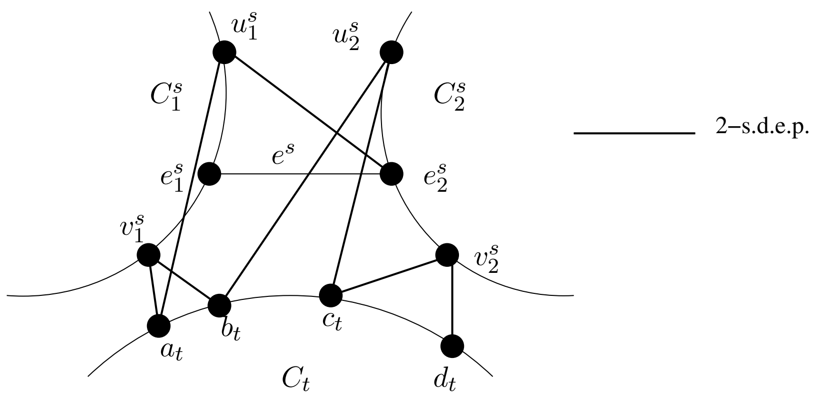

Figure 2: The 2-s.d.e.p. W_t^s of claim 3.

Let $r' \geqslant 2$ and consider, for $t = 1, \ldots, r'$, the (residual) cycles C_t. All edges between these cycles are of distance 2. If we denote by a_t, b_t, c_t and d_t four vertices of C_t, the path $\{a_1, \ldots, a_t, \ldots, a_{r'}, b_1, \ldots, b_t, \ldots, b_{r'}, c_1, \ldots, c_t, \ldots, c_{r'}, d_1, \ldots, d_t, \ldots, d_{r'}\}$ is a 2-s.d.e.p. of size $4r' - 1$ and the following claim holds.

Claim 4. *If $r' \geqslant 2$, then there exists a 2-s.d.e.p. $W^{r'}$ of size $4r' - 1$ alternating vertices of cycles C_t, $t = 1, \ldots, r'$.*

Lemma 5. *If $q \geqslant 1$, or if $[(C_0 \neq E_2)$ and $(r \neq 1)]$, then $\delta_{\texttt{TSP12}}(K_n) \geqslant 3/4$.*

Proof. Consider M as expressed in (2). Let us denote by W' a 2-s.e.d.p. on $V \setminus V(C_0)$. From claims 1, 2, 3 and 4:

$$W' = \begin{cases} \bigcup_{s=1}^{r'} W_s^s \bigcup_{s=r'+1}^{q} W^s & |W'| = 8r' + 4(q - r') = 4(q + r') & q \geqslant r' \\ \bigcup_{s=1}^{q} W^s \bigcup W^{r'} \bigcup \{\gamma\} & |W'| = 4q + 4r' - 1 + 1 = 4(q + r') & r' > q > 0 \\ W^{r'} & |W'| = 4r' - 1 = 4(q + r') - 1 & r' \geqslant 2, q = 0 \end{cases} \quad (3)$$

In (3), γ draws an edge linking a vertex of degree 1 with respect to $W^{r'}$ to a vertex of degree 1 in $V(W')$. This last vertex can belong either to $V(C_3^1)$ if $q' \geqslant 1$, or to V_q otherwise. By the optimality of M_H and M_B, γ is a 2-edge. For the first line of (3), remark that if $q = r'$, then W' is given by $\cup_{s=1}^{r'} W_s^s$, and claims 2 and 3 conclude $W' = 8r' = 4(r' + q)$; otherwise $(q > r')$, claims 1, 2 and 3 conclude the first line of (3). For the second line $(r' > q > 0)$, since r' and q are integers, we have $r' \geqslant 2$. Then, by claim 4, $|W^{r'}| \geqslant 4r' - 1$ and the expression of the second line follows from the fact that γ is a 2-edge. For the third line of (3), claim 4 gives immediately the result. Hence, in any case, W' is a 2-s.d.e.p. verifying $|W'| = 4(q + r') - 1$ if $q = 0$, $|W'| = 4(q + r')$ otherwise.

We now construct a a 2-s.e.d.p. W_0 on $V(C_0)$. Let $g_0 = u_0v_0$ be the edge removed from C_0 during the execution of step `(4)` of algorithm `TSP12`. If $C_0 \neq E2$, then g_0 has been chosen in such a way that one of its endpoints, say v_0, is adjacent in C_0 to an 1-edge. Let γ' be an edge linking v_0 to a vertex of degree 1 with respect to W' (such a vertex exists since W' is acyclic). By the 2-minimality of M, $d(\gamma') = 2$. Set

$$W_0 = \begin{cases} E2 \setminus \{g_0\} \cup \{\gamma'\} & |W_0| = d(M) - n & C_0 = E2 \\ E2 \cup \{\gamma'\} & |W_0| = d(M) - n + 1 & \text{otherwise} \end{cases}$$

Setting finally $W = W' \cup W_0$, one obtains the 2-s.d.e.p. claimed at the beginning of section 3.3. We have $d(W) = 4(q+r') - \mathbf{1}_{q=0} + d(M) - n + \mathbf{1}_{C_0 \neq E2} \geqslant d(M) - n + 4(q+r')$, the last inequality holding because of the hypothesis $q = 0 \Rightarrow C_0 \neq E_2$ made just before lemma 2. Any completion of W in a Hamiltonian cycle of K_n (as in step `(5)` of algorithm `TSP12`) would produce a tour T_w of total distance $d(T_w) \geqslant d(M) + 4(q + r')$. So, $\omega(K_n) \geqslant d(M) + 4(q + r')$, and combining this expression together with lemmata 2 and 3, the differential ratio 3/4 is immediately concluded. ∎

Lemma 6. *If $q = 0$ and $r = 1$ and $C_0 \neq E_2$, then $\delta_{\texttt{TSP12}}(K_n) \geqslant 3/4$.*

Proof. Recall that, by the optimality of M_H, if $q = 0$ and $r = 1$, all edges linking cycles C_0 and C_1 are of distance 2. Let u_0, u_1 and u_2 be three vertices of C_0 such that $d(u_0, u_1) = 1$ and $d(u_1, u_2) = 2$. We denote by u_2 a neighbor of u_1 with respect to C_0, and by $u_3 \neq u_1$ the neighbor of u_2 in C_0. Let a, b, c and d be any four vertices of C_1, and let W_p and W_m be the sets $\{au_1, u_1b, bu_2, u_2c\}$ and $\{u_1u_2\}$, respectively.

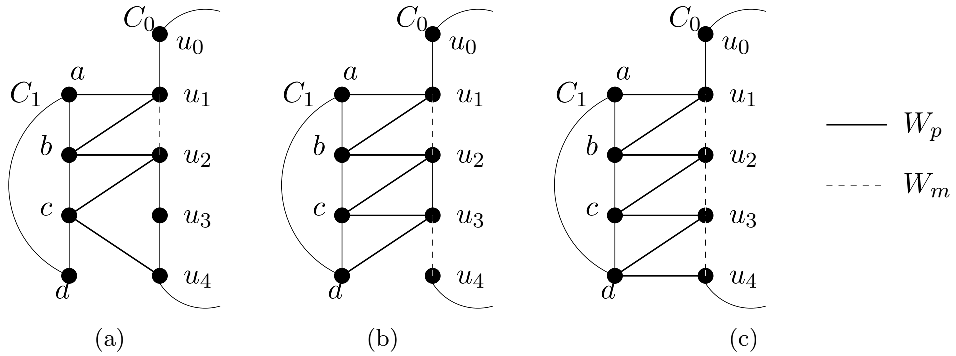

Figure 3:

If u_3 is adjacent to two 1-edges in C_0, i.e., $d(u_2, u_3) = d(u_3, u_4) = 1$, then set $W_p = W_p \cup \{cu_4\}$ (figure 3(a)).

If only one of u_2u_3 and u_3u_4 is of distance 2, then set $W_p = W_p \cup \{cu_3, u_3d\}$ and $W_m = W_m \cup \{\text{argmax}\{d(u_2, u_3), d(u_3, u_4)\}\}$. In figure 3(b), sets W_p and W_m are shown for the case where $\text{argmax}\{d(u_2, u_3), d(u_3, u_4)\} = u_3u_4$, i.e., $d(u_3, u_4) = 2$ and $d(u_2, u_3) = 1$.

Finally, if u_2u_3 and u_3u_4 are both of distance 2 ($d(u_2, u_3) = d(u_3, u_4) = 2$), then set $W_p = W_p \cup \{cu_3, u_3d, du_4\}$ and $W_m = W_m \cup \{u_2u_3, u_3u_4\}$ (figure 3(c)).

In any of the above cases, we consider the 2-s.d.e.p. $W = E2 \cup W_p \setminus W_m$ with $|W| = d(M) - n + 4 = d(M) - n + 4r'$. We then have $\omega(K_n) \geqslant d(M) + 4r'$, and combining it with lemmata 2 and 3, the differential ratio 3/4 is concluded. ∎

In all, combining lemmata 2, 3, 4 and 6, the following theorem can be immediately proved.

Theorem 2. *min_TSP12 is approximable within differential-approximation ratio 3/4.*

Theorems 1 and 2 induce the following corollary.

Corollary 1. *min_TSP12, max_TSP12, min_TSPab and max_TSPab are approximable within differential-approximation ratio 3/4.*

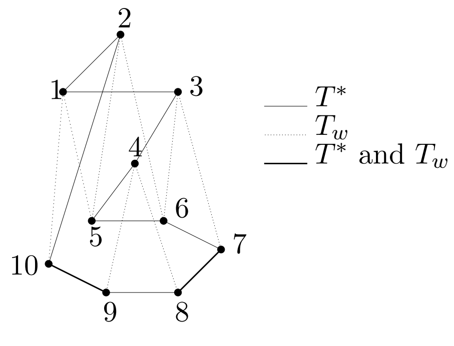

Figure 4: Tightness of the `TSP12` approximation ratio.

Proposition 3. *Ratio 3/4 is tight for* `TSP12`

Proof. Consider two cliques and number their vertices by $\{1, \ldots, 4\}$ and by $\{5, 6, \ldots, n+8\}$, respectively. Edges of both cliques have all distance 1. Cross-edges ij, $i = 1, 3$, $j = 5, \ldots, n+8$,

	k	TSP12	The algorithm of [11]
Differential ratio	3	0.931100364	0.846702091
	4	0.9000002	0.833333
	5	0.920289696	0.833333
	6	0.9222222	0.833333
Standard ratio	3	0.923350955	0.87013
	4	0.9094018	0.857143
	5	0.92646313	0.857143
	6	0.928178	0.857143

Table 1: A limited comparison between TSP12 and the algorithm of [11] on some worst-case instances of the latter.

are all of distance 2, while every other cross-edge is of distance 1. Unraveling of TSP12 will produce: $T = \{1, 2, 3, 4, 5, 6, \ldots, n + 7, n + 8, 1\}$ (cycle-patching on edges $(1, 4)$ and $(5, n + 8)$), while $T_w = \{1, 5, 2, 6, 3, 7, 4, 8, 9 \ldots, n+7, n+8, 1\}$ (using 2-edges $(1, 5)$, $(6, 3)$, $(3, 7)$ and $(n+8, 1)$) and $T^* = \{1, 2, n + 8, n + 7, \ldots, 5, 4, 3, 1\}$ (using 1-edges $(4, 5)$ and $(2, n + 8)$). Consequently, $\delta_{\texttt{TSP12}}(K_{n+8}) = 3/4$, q.e.d. ∎

In figure 4, the tours T^* and T_w of proposition 3 are shown for $n = 2$. We assume $T = \{1, \ldots, 10, 1\}$.

Let us note that the differential approximation ratio of the 7/6-algorithm of [11], when running on K_{n+8}, is also 3/4. The authors of [11] exhibit a family of worst-case instances for their algorithm: one has k cycles of length 4 arranged around a cycle of length $2k$. We have performed a limited comparative study between their algorithm and ours, for $k = 3, 4, 5, 6$ (on 24 graphs). The average differential and standard approximation ratios for the two algorithms are presented in table 1.

Proposition 4. *min_TSPab is approximable within standard-approximation ratio* $\rho \leqslant (1 + ((b - a)/(4a))$. *This ratio tends to* ∞ *when* $a = o(b)$.

Proof. Revisit corollary 1. Differential ratio 3/4 for min_TSPab implies $\lambda(K_n)/\beta(K_n) \leqslant (3/4) + (\omega(K_n)/(4\beta(K_n)))$. Using $\omega(K_n) \leqslant bn$ and $\beta(K_n) \geqslant an$, some easy algebra gives the result claimed. ∎

Theorem 3. *min_ and max_TSPab and min_ and max_TSP12 are inapproximable within differential-ratio of at least* $742/743 + \epsilon$, $\forall \epsilon > 0$, *unless* ***P=NP***.

Proof. Consider min_TSP12. Using $n \leqslant \beta(K_n) \leqslant \omega(K_n) \leqslant 2n$, one can see that approximation of min_TSP12 within $\delta = 1-\epsilon$ implies its approximation within $\rho = 2-(1-\epsilon) = 1+\epsilon$, $0 \leqslant \epsilon \leqslant 1$. Then, the inapproximability bound $(743/742 - \epsilon)$ of [7] for min_TSP12 together with theorem 1 conclude the proof. ∎

4 An improvement of the standard ratio for the maximum traveling salesman with distances 1 and 2

We propose in this section a non-trivial improvement of the standard-approximation ratio for max_TSP12, by proving the following theorem.

Theorem 4. *max_TSP12 is polynomially approximable within standard-approximation ratio bounded below by 7/8.*

Proof. Combining, as in the proof of proposition 4, expressions $\delta_{\max\_\mathrm{TSP12}} \geqslant 3/4$, $\omega_{\max}(K_n) \geqslant an$ and $\beta_{\max}(K_n) \leqslant bn$, one deduces $\rho_{\max\_\mathrm{TSP12}} \geqslant (3/4) + (a/4b)$. Setting $a = 1$ and $b = 2$, the result claimed follows immediately. ∎

Remark 2. Consider now the following simple way to use the algorithm of [11] in order to solve max_TSP12 on a graph K_n with edge-distances 1 and 2. The graph $\bar{K}_n$ in the first line of the algorithm `MTSPALG` just above is such that the distance of an edge e of $\bar{K}_n$ is 1 if e has distance 2 in K_n, and 2 if e has distance 1 in K_n.

```
BEGIN *MTSPALG*
      construct K̄n;
      call the algorithm of [11] to compute a tour Tmin(K̄n);
      OUTPUT Tmax(Kn) ← Tmin(K̄n);
END.
```

Let us denote by `A` the 7/6-algorithm of [11] called in the second line of `MTSPALG`. Then, by theorem 1, $\beta(K_n) = 3n - \beta(\bar{K}_n)$ and $\lambda_{\mathtt{MTSPALG}}(K_n) = 3n - \lambda_{\mathtt{A}}(\bar{K}_n)$, where $\beta(K_n)$ is the optimal value for max_TSP12 on K_n and $\beta(\bar{K}_n)$ is the optimal value for min_TSP12 on $\bar{K}_n$. Then, $\lambda_{\mathtt{A}}(\bar{K}_n)/\beta(\bar{K}_n) \leqslant 7/6$, together with $\beta(\bar{K}_n) \geqslant n$ imply $\lambda_{\mathtt{MTSPALG}}(K_n)/\beta(K_n) \geqslant 2/3$. ∎

Note finally that standard-approximation ratio 7/8 can be obtained by the following direct method.

```
BEGIN *max_TSP12*
      find a triangle-free 2-matching M = {C1, C2, ...};
      FOR all Ci DO delete a minimum-distance edge from Ci OD
      properly link the remaining paths to obtain a Hamiltonian cycle T;
      OUTPUT T;
END. *max_TSP12*
```

Let p be the number of cycles of M where 2-edges have been removed during the `FOR`-loop of algorithm `max_TSP12`. Then, $\lambda_{\mathtt{max\_TSP12}}(K_n) \geqslant d(M) - p$, $\beta(K_n) \leqslant d(M)$, and since M is triangle-free, $d(M) \geqslant 8p$. Consequently, $\lambda_{\mathtt{max\_TSP12}}(K_n)/\beta_{\max}(K_n) \geqslant 7/8$.

The above result, can be extended to the case of max_TSPab if we consider that here $\lambda_{\mathtt{max\_TSP12}}(K_n) \geqslant d(M) - p(b-a)$, $\beta(K_n) \leqslant d(M)$ and $d(M) \geqslant 4bp$. Hence the following corollary holds and concludes the paper.

Corollary 2. *max_TSPab is polynomially approximable within standard-approximation ratio bounded below by* $(3/4) + (a/(4b))$.

References

[1] A. Aiello, E. Burattini, M. Furnari, A. Massarotti, and F. Ventriglia. Computational complexity: the problem of approximation. In C. M. S. J. Bolyai, editor, *Algebra, combinatorics, and logic in computer science*, volume I, pages 51–62, New York, 1986. North-Holland.

[2] G. Ausiello, A. D'Atri, and M. Protasi. Structure preserving reductions among convex optimization problems. *J. Comput. System Sci.*, 21:136–153, 1980.

[3] G. Ausiello, A. Marchetti-Spaccamela, and M. Protasi. Towards a unified approach for the classification of NP-complete optimization problems. *Theoret. Comput. Sci.*, 12:83–96, 1980.

[4] M. Bellare and P. Rogaway. The complexity of approximating a nonlinear program. *Math. Programming*, 69:429–441, 1995.

[5] M. Demange, P. Grisoni, and V. T. Paschos. Differential approximation algorithms for some combinatorial optimization problems. *Theoret. Comput. Sci.*, 209:107–122, 1998.

[6] M. Demange and V. T. Paschos. On an approximation measure founded on the links between optimization and polynomial approximation theory. *Theoret. Comput. Sci.*, 158:117–141, 1996.

[7] L. Engebretsen and M. Karpinski. Approximation hardness of TSP with bounded metrics. Report 89, Electr. Colloq. Computational Comp., 2000.

[8] M. R. Garey and D. S. Johnson. *Computers and intractability. A guide to the theory of NP-completeness*. W. H. Freeman, San Francisco, 1979.

[9] D. B. Hartvigsen. *Extensions of matching theory*. PhD thesis, Carnegie-Mellon University, 1984.

[10] J. Monnot, V. T. Paschos, and S. Toulouse. Differential approximation results for the traveling salesman problem. Cahier du LAMSADE 172, LAMSADE, Universit Paris-Dauphine, 2000.

[11] C. H. Papadimitriou and M. Yannakakis. The traveling salesman problem with distances one and two. *Math. Oper. Res.*, 18:1–11, 1993.

[12] A. I. Serdyukov. An algorithm with an estimate for the traveling salesman problem of the maximum. *Upravlyaemye Sistemy*, 25:80–86, 1984.

[13] S. A. Vavasis. Approximation algorithms for indefinite quadratic programming. *Math. Programming*, 57:279–311, 1992.

[14] E. Zemel. Measuring the quality of approximate solutions to zero-one programming problems. *Math. Oper. Res.*, 6:319–332, 1981.