A note on homomorphisms of Kneser hypergraphs*

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Abstract

Let n, k, r be positive integers, with $n \geq kr$. The r-uniform Kneser hypergraph $KG^r(n,k)$ has as vertex set the set of all k-subsets of the set $\{1,\ldots,n\}$ and its (hyper) edges are formed by the r-tuples of pairwise disjoint k-subsets of the set $\{1,\ldots,n\}$. In this paper, we give conditions for the existence of homomorphisms between uniform Kneser hypergraphs.

Keywords: Kneser Hypergraph, Hypergraph Homomorphism, Hypergraph Coloring.

1 Introduction and preliminaries

A *hypergraph* $\mathcal{H}$ is an ordered pair $(\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, where $\mathcal{V}(\mathcal{H})$ (the *vertex set*) is a finite set and $\mathcal{E}(\mathcal{H})$ (the *edge set*) is a family of distinct non-empty subsets of $\mathcal{V}(\mathcal{H})$. If every (hyper) edge in $\mathcal{E}(\mathcal{H})$ has size r, then $\mathcal{H}$ is called *r-uniform*. Notice that a (simple) graph is a 2-uniform hypergraph. Let A, B be two finite sets and let $\phi : A \to B$ be a mapping from A to B. The *extension of* ϕ, that we denote by $\hat{\phi}$, is a mapping from 2^A to 2^B defined by $\hat{\phi}(S) = \cup_{a\in S}\{\phi(a)\}$, for any subset $S \subseteq A$.

Let $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ and $\mathcal{H} = (\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ be two hypergraphs. A mapping $\phi : \mathcal{V}(\mathcal{G}) \to \mathcal{V}(\mathcal{H})$ is called a *homomorphism* from $\mathcal{G}$ to $\mathcal{H}$ if, for any edge $e \in \mathcal{E}(\mathcal{G})$, we have that $\hat{\phi}(e) \in \mathcal{E}(\mathcal{H})$. If there is a homomorphism ϕ from $\mathcal{G}$ to $\mathcal{H}$, we will write $\mathcal{G} \to \mathcal{H}$, and also introduce ϕ writing $\phi : \mathcal{G} \to \mathcal{H}$. An *automorphism* of a (hyper)graph $\mathcal{G}$ is an injective homomorphism from $\mathcal{G}$ to himself. The set of all automorphisms of a (hyper)graph $\mathcal{G}$ forms a group structure which is denoted by $\mathrm{Aut}(\mathcal{G})$.

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For any positive integer t, let $[t]$ denote the set $\{1, 2, \ldots, t\}$. A *t-coloring* of a hypergraph $\mathcal{H}$ is a coloring $f : \mathcal{V}(\mathcal{H}) \to [t]$ of the vertex set with t colors such that there is no monochromatic edge. The minimum t such that there exists a t-coloring for hypergraph $\mathcal{H}$ is called its *chromatic number*, and it is denoted by $\chi(\mathcal{H})$.

For any positive integers n, k, let $\binom{[n]}{k}$ be the set of k-subsets of $[n]$. The Kneser hypergraph $KG^r(n,k)$ is the r-uniform hypergraph whose vertex set is $\binom{[n]}{k}$ and whose (hyper) edges are formed by the r-tuples of pairwise disjoint k-subsets of $[n]$.

Concerning the study of homomorphisms between 2-uniform Kneser hypergraphs, the most known and useful results are the following:

Lemma 1 (Stahl [5]). *Let n, k be positive integers with $n \geq 2k$. Then, there is a homomorphism $KG^2(n+2, k+1) \to KG^2(n,k)$.*

Notice that if $H \to KG^2(n_1, k_1)$ and $H \to KG^2(n_2, k_2)$, then $H \to KG^2(n_1 + n_2, k_1 + k_2)$. Therefore, by using the Stahl's homomorphism we can deduce that $KG^2(n,k) \to KG^2(tn - 2s, tk - s)$ for any positive integer t and any $s \in [k-1]$.

Lemma 2 (Godsil and Roy [3]). *Let $n/k = w/s > 2$. Then, there is a homomorphism $KG^2(n,k) \to KG^2(w,s)$ if and only if k divides s.*

Lemma 3 (Godsil and Roy [3]). *Suppose there is a homomorphism $KG^2(n,k) \to KG^2(w,s)$. If $s\binom{n}{k} > n\binom{n-1}{k-1} + (w-n)h_{n,k}$, then there is a homomorphism $KG^2(n-1,k) \to KG^2(w-2,s)$, where $h_{n,k} = 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$.*

The chromatic number of r-uniform Kneser hypergraphs has been completely determined. In a famous paper, Lovász [4] proved that $\chi(KG^2(n,k))$ is equal to $n - 2k + 2$. Later, this result has been extended to r-uniform Kneser hypergraphs by Alon, Frankl and Lovász [1] who showed that $\chi(KG^r(n,k)) = \left\lceil \frac{n-(k-1).r}{r-1} \right\rceil$ for $n \geq kr$.

As far as we know, there are no results concerning the study of homomorphisms between r-uniform Kneser hypergraphs for $r > 2$. In this paper, we give some necessary and sufficient conditions for the existence of homomorphisms between Kneser hypergraphs. The paper is organized as follows: in Section 2, we start our study by characterizing the existence of homomorphisms between two r-uniform Kneser hypergraphs. The study of homomorphisms between two Kneser hypergraphs formed by hyperedges of different size is done in Sections 3 and 4. In Section 3, we study the homomorphisms from $KG^r(n,1)$ to any other r'-uniform Kneser hypergraph. In Section 4, we present results for the more general case of homomorphisms from r-uniform Kneser hypergraphs to r'-uniform Kneser hypergraphs. Finally, in Section 5, we discuss some applications of our results to rainbow colorings of Kneser hypergraphs.

2 Homomorphism between two r-uniform Kneser hypergraphs

In this section, we characterize the existence of homomorphisms between two r-uniform Kneser hypergraphs in terms of the existence of homomorphisms between 2-uniform Kneser hypergraphs.

Theorem 1. *Let r, n_1, k_1, n_2, k_2 be positive integers, with $n_i \geq rk_i$, for $i = 1, 2$, and with $r \geq 3$. There is a homomorphism from $KG^r(n_1, k_1)$ to $KG^r(n_2, k_2)$ if and only if there is a homomorphism from $KG^2(n_1, k_1)$ to $KG^2(n_2, k_2)$.*

Proof. Assume there is a homomorphism $\phi : KG^r(n_1, k_1) \to KG^r(n_2, k_2)$. Let A and B be a pair of adjacent vertices in $KG^2(n_1, k_1)$. As $n_1 \geq rk_1$ and $r \geq 3$, there exists a set of $r-2$ pairwise disjoint elements $\{C_1, \ldots, C_{r-2}\}$ of $\binom{[n_1]\setminus(A\cup B)}{k_1}$. Thus, the set $e = \{A, B, C_1, \ldots, C_{r-2}\}$ is an edge of $KG^r(n_1, k_1)$ and therefore, by hypothesis, the set $\hat{\phi}(e) = \{\phi(A), \phi(B), \phi(C_1), \ldots, \phi(C_{r-2})\}$ is an edge of $KG^r(n_2, k_2)$, which implies that $\phi(A) \cap \phi(B) = \emptyset$. Therefore, ϕ is a homomorphism from $KG^2(n_1, k_1)$ to $KG^2(n_2, k_2)$.

Conversely, let ϕ be a homomorphism from $KG^2(n_1, k_1)$ to $KG^2(n_2, k_2)$. By hypothesis, for any pair of vertices A, B in $\binom{[n_1]}{k_1}$ such that $A \cap B = \emptyset$, we have that $\phi(A) \cap \phi(B) = \emptyset$. Therefore, by definition of r-uniform Kneser hypergraphs, we have that each edge of $KG^r(n_1, k_1)$ is mapped by ϕ to an edge of $KG^r(n_2, k_2)$, which proves that ϕ is also a homomorphism from $KG^r(n_1, k_1)$ to $KG^r(n_2, k_2)$. $\square$

Remark 1. *Let $\mathcal{H}$ be an r_1-uniform hypergraph and $\mathcal{G}$ be an r_2-uniform hypergraph. If there is a homomorphism from $\mathcal{H}$ to $\mathcal{G}$ then $r_1 \geq r_2$.*

In fact, notice that if $\phi : \mathcal{H} \to \mathcal{G}$ is a homomorphism and $e = \{v_1, \ldots, v_{r_1}\}$ is an edge of $\mathcal{H}$, then $\hat{\phi}(e) = \{\phi(v_1), \ldots, \phi(v_{r_1})\}$ is an edge of $\mathcal{G}$ and therefore $r_2 \leq r_1$.

3 The case $k_1 = 1$

In Theorem 2 we completely characterize the existence of homomorphisms between $KG^{r_1}(n_1, 1)$ and other r-uniform Kneser graphs. First, in Lemma 4 we show that such homomorphism should send all vertices of $KG^{r_1}(n_1, 1)$ to a single edge in the image. Then Example 1 shows this condition is not sufficient. The rest of the section is then devoted to characterize such homomorphisms.

Lemma 4. *Let n_1, n_2, r_1, r_2, k_2 be positive integers with $n_1 \geq r_1$, $n_2 \geq r_2k_2$, and $r_1 > r_2$. Let ϕ be a homomorphism from $KG^{r_1}(n_1, 1)$ to $KG^{r_2}(n_2, k_2)$. Then, for any pair of (hyper)edges e_1, e_2 in $KG^{r_1}(n_1, 1)$, we must have that $\phi(e_1) = \phi(e_2)$.*

Proof. Let $e_1 = \{u_1, \ldots, u_{r_1}\}$ be an edge of $KG^{r_1}(n_1, 1)$. As ϕ is a homomorphism, then $\hat{\phi}(e_1) = e$, where $e = \{v_1, \ldots, v_{r_2}\}$ is an edge of $KG^{r_2}(n_2, k_2)$. By sake of contradiction, assume there is a vertex $x \in [n_1]\setminus e_1$ such that $y = \phi(x) \notin e$. As $r_1 > r_2$, there exists $v_i \in e$ such that $|\phi^{-1}(v_i) \cap e_1| > 1$. Let $u \in \phi^{-1}(v_i) \cap e_1$. Notice that $e' = (e_1 \setminus \{u\}) \cup \{x\}$ is an edge of $KG^{r_1}(n_1, 1)$, and $e' \neq e_1$. However, $\phi(e') = e \cup \{y\}$ which is not an edge of $KG^{r_2}(n_2, k_2)$, contradicting the fact that ϕ is a hypergraph homomorphism. $\square$

By Lemma 4, if ϕ is a homomorphism from $KG^{r_1}(n_1, 1)$ to $KG^{r_2}(n_2, k_2)$, then $\hat{\phi}([n_1])$ is an edge of $KG^{r_2}(n_2, k_2)$. However, this fact is not a sufficient condition for determining whether there exists or not a homomorphism between $KG^{r_1}(n_1, 1)$ and $KG^{r_2}(n_2, k_2)$ as Example 1 shows.

Example 1. *Consider $n_1 \geq 5$, $r_1 = 3$, $k_2 = 1$, $r_2 = 2$, and $n_2 \geq 2$. We show that there is not homomorphism from $KG^3(n_1, 1)$ to $KG^2(n_2, 1)$. By sake of contradiction assume $\phi : KG^3(5, 1) \to KG^2(n_2, 1)$. From Lemma 4 we have $\hat{\phi}([n_1]) = e = \{v_1, v_2\}$, where e is an edge of $KG^2(n_2, 1)$. As $\{\phi^{-1}(v_1), \phi^{-1}(v_2)\}$ is a partition of $[n_1]$, w.l.o.g. we can assume that $|\phi^{-1}(v_1)| \geq 3$. Thus $\phi^{-1}(v_1)$ contains an edge e' of $KG^3(n_1, 1)$, which is a contradiction as we have $\phi(e') = \{v_1\}$ which is not and edge of $KG^2(n_2, 1)$.*

Therefore, there exists no homomorphism between $KG^3(n_1, 1)$ and $KG^2(n_2, 1)$. In particular, notice that $\chi(KG^3(5, 1)) = \chi(KG^2(3, 1)) = 3$ but $KG^3(5, 1) \not\to KG^2(3, 1)$.

Given $\phi : KG^{r_1}(n_1,1) \to KG^{r_2}(n_2,k_2)$, using Lemma 4 one can define a partition $n_1 = a_1 + \cdots + a_{r_2}$ of n_1 into r_2 positive parts, where each part corresponds to the size of the pre-image under ϕ of a vertex in $\hat{\phi}([n_1])$. We call such partition the *type* of ϕ. In fact, notice that by Lemma 4, as ϕ is a homomorphism, then all vertices in $KG^{r_1}(n_1,1)$ are mapped by ϕ to one hyperedge $e = \{v_1, v_2, \cdots, v_{r_2}\}$ in $\mathcal{E}(KG^{r_2}(n_2,k_2))$ and thus, $\hat{\phi}([n_1]) = e$. Therefore, the type of ϕ is the r_2-partition $(a_1, a_2, \cdots, a_{r_2})$ of n_1, where $a_i = |\phi^{-1}(v_i)|$ for $i = 1, 2, \cdots, r_2$.

Definition 1. *An r-partition of n is a vector $\mathbf{a} = (a_1, \ldots, a_r)$ of size r with $n = a_1 + \cdots + a_r$ and $0 < a_1 \le a_2 \le \cdots \le a_r$.*

As Example 1 shows, not every r_2-partition of n_1 is the type of a homomorphism from $KG^{r_1}(n_1,1)$ to $KG^{r_2}(n_2,k_2)$. In Lemma 5, we give a characterization of when a partition is the type of a homomorphism. In Lemma 6, we show that modulo automorphisms of the two hypergraphs, the type characterizes the homomorphism. Lemmas 5 and 6 completely characterize the set of all homomorphisms from $KG^{r_1}(n_1,1)$ to $KG^{r_2}(n_2,k_2)$ for any positive integers n_1, n_2, r_1, r_2, k_2 with $n_1 \ge r_1$, $n_2 \ge r_2 k_2$, and $r_1 > r_2$.

Lemma 5. *Let $\mathbf{a}$ be an r_2-partition of n_1. Then, $\mathbf{a}$ is the type of a homomorphism from $KG^{r_1}(n_1,1)$ to $KG^{r_2}(n_2,k_2)$ if and only if $a_1 + r_1 > n_1$.*

Proof. First, assume $\mathbf{a}$ is the type of $\phi : KG^{r_1}(n_1,1) \to KG^{r_2}(n_2,k_2)$. Then each a_i is the size of the set $\phi^{-1}(v_i)$ where v_i is a vertex in $\hat{\phi}([n_1])$. Let $S = [n_1] \setminus \phi^{-1}(v_1)$. If $|S| \ge r_1$, taking $S' \subseteq S$ of size r_1, we have that S' is an edge of $KG^{r_1}(n_1,1)$, but $\hat{\phi}(S') \subseteq \hat{\phi}([n_1]) \setminus \{v_1\}$ which is not an edge of $KG^{r_2}(n_2,k_2)$. Therefore $r_1 > |S| = n_1 - a_1$. Now, assume $\mathbf{a}$ is such that $a_1 + r_1 > n_1$. Let $e = \{v_1, \ldots, v_{r_2}\}$ be a fixed edge of $KG^{r_2}(n_2,k_2)$. For each $i \in [n_1]$, define $\phi(i) = v_j$ where $j \in [r_2]$ is the index such that $a_1 + \cdots + a_{j-1} < i \le a_1 + \cdots + a_{j-1} + a_j$. Clearly, ϕ is a map from $[n_1]$ to $\binom{[n_2]}{k_2}$ such that $\hat{\phi}([n_1]) = e$. If ϕ is not a homomorphism from $KG^{r_1}(n_1,1)$ to $KG^{r_2}(n_2,k_2)$, then there is an edge e_1 of $KG^{r_1}(n_1,1)$ and $j \in [r_2]$ such that $v_j \notin \hat{\phi}(e_1)$. Then $|e_1| \le |\hat{\phi}^{-1}(\hat{\phi}(e_1))| \le n_1 - a_j \le n_1 - a_1 < r_1$ which is a contradiction. □

Lemma 6. *Let $n_1, n_2, r_1, r_2, k_1, k_2$ be positive integers with $k_1 = 1$, $n_1 \ge r_1$, $n_2 \ge r_2 k_2$, and $r_1 > r_2$. Let ϕ_1 and ϕ_2 be two homomorphisms from $KG^{r_1}(n_1,k_1) \to KG^{r_2}(n_2,k_2)$ with types a^1 and a^2, respectively. Then $a^1 = a^2$ if and only if there are α_i in $Aut(KG^{r_i}(n_i,k_i))$ for $i \in \{1,2\}$ such that $\phi_1\alpha_1 = \alpha_2\phi_2$.*

Proof. Let e_1 and e_2 be the edges of $KG^{r_2}(n_2,k_2)$ such that $\phi_i(e) = e_i$ for any edge $e \in \mathcal{E}(KG^{r_1}(n_1,1))$ and $i \in \{1,2\}$. First, consider that $a^1 = a^2$. Now, for $v \in e_1$, define $\alpha_2(v) = u$ where $u \in e_2$ and $|\phi_2^{-1}(v)| = |\phi_1^{-1}(u)|$ in such way that $\alpha_2(v) \ne \alpha_2(v')$ for $v, v' \in e_1$; complete the definition of α_2 by using any injective function from $\mathcal{V}(KG^{r_2}(n_2,k_2)) \setminus e_1$ to $\mathcal{V}(KG^{r_2}(n_2,k_2)) \setminus e_2$. Since $a^1 = a^2$, α_2 is well defined. Next, define α_1 by using, for every $v \in e_1$, an injective function from $\phi_2^{-1}(v)$ and $\phi_1^{-1}(\alpha_2(v))$. Notice that $\phi_1\alpha_1 = \alpha_2\phi_2$.

Conversely, assume that there are α_i in $\mathrm{Aut}(KG^{r_i}(n_i,k_i))$ for $i \in \{1,2\}$ such that $\phi_1\alpha_1 = \alpha_2\phi_2$. It is clear that α_2 restricted to e_1 is an injective function with image e_2. Notice that every vertex $v \in e_1$ contributes with value $|\phi_2^{-1}(v)|$ to compose the type a^2 and the contribution of $\alpha_2(v) \in e_2$ to compose a^1 is $|\phi_1^{-1}(\alpha_2(v))|$. Since $\phi_1\alpha_1 = \alpha_2\phi_2$, it holds $|\phi_2^{-1}(v)| = |\phi_1^{-1}(\alpha_2(v))|$ and therefore $a^1 = a^2$. □

The next result shows necessary and sufficient conditions for determining the existence of a homomorphism between two hypergraphs $KG^{r_1}(n_1,1)$ and $KG^{r_2}(n_2,k_2)$, with $r_1 \ne r_2$.

Theorem 2. *Let n_1, n_2, r_1, r_2, k_2 be positive integers with $n_1 \geq r_1$, $n_2 \geq r_2 k_2$, and $r_1 > r_2$. Then the following are equivalent:*

(i) There exists $\phi : KG^{r_1}(n_1, 1) \to KG^{r_2}(n_2, k_2)$

(ii) There exists an r_2-partition $\mathbf{a}$ of $[n_1]$ such that $a_1 + r_1 > n_1$

(iii) $n_1 - \lfloor \frac{n_1}{r_2} \rfloor < r_1$

(iv) $r_2 = 1$ or $n_1 \leq \lfloor \frac{r_2(r_1-1)}{r_2-1} \rfloor$.

Proof. That (i) and (ii) are equivalent follows from Lemma 5. Notice that if (ii) holds, then $a_1 \leq (a_1 + \cdots + a_{r_2})/r_2 = n_1/r_2$ and thus (iii) follows. Also, if (iii) holds, let $s = n_1 - r_2 \lfloor \frac{n_1}{r_2} \rfloor$. Then $0 \leq s < r_2$. Define $a_i = \lfloor \frac{n_1}{r_2} \rfloor$ for $i = 1, \ldots, r_2 - s$ and $a_i = \lfloor \frac{n_1}{r_2} \rfloor + 1$ for $i = r_2 - s + 1, \ldots, r_2$. Then $\mathbf{a}$ is an r_2-partition satisfying (ii).

Now, to show that (iii) and (iv) are equivalent, notice first that the case $r_2 = 1$ is trivial. Therefore we assume $r_2 > 1$. Let $N = \lfloor \frac{r_2(r_1-1)}{r_2-1} \rfloor$. Notice that if $n_1 \geq r_1$ satisfies (iii) (resp. (iv)) and $r_1 \leq n_1' \leq n_1$, then n_1' also satisfies (iii) (resp. (iv)). Thus to show that (iii) and (iv) are equivalent it is enough to show that (iii) holds for $n_1 = N$ and does not hold for $n_1 = N + 1$. From the definition of N, we have that $N > \frac{r_2(r_1-1)}{r_2-1} - 1$. Thus,

$$N - \left\lfloor \frac{N}{r_2} \right\rfloor < \left\lfloor \frac{r_2(r_1-1)}{r_2-1} \right\rfloor - \left\lfloor \frac{r_1-1}{r_2-1} - \frac{1}{r_2} \right\rfloor = r_1 - 1 + \left\lfloor \frac{r_1-1}{r_2-1} \right\rfloor - \left\lfloor \frac{r_1-1}{r_2-1} - \frac{1}{r_2} \right\rfloor \leq r_1$$

as $r_2 \geq 2$. Also, $N \leq \frac{r_2(r_1-1)}{r_2-1}$ and thus

$$N + 1 - \left\lfloor \frac{N+1}{r_2} \right\rfloor \geq \left\lfloor \frac{r_2(r_1-1)}{r_2-1} \right\rfloor + 1 - \left\lfloor \frac{r_1-1}{r_2-1} + \frac{1}{r_2} \right\rfloor = r_1 + \left\lfloor \frac{r_1-1}{r_2-1} \right\rfloor - \left\lfloor \frac{r_1-1}{r_2-1} + \frac{1}{r_2} \right\rfloor. \quad (1)$$

Using that for any positive real x and positive integer n we have $\lfloor x \rfloor n \leq \lfloor nx \rfloor$, we obtain that $\lfloor \frac{r_1-1}{r_2-1} + \frac{1}{r_2} \rfloor (r_2 - 1) \leq \lfloor r_1 - 1 + \frac{r_2-1}{r_2} \rfloor = r_1 - 1$. Thus, $\lfloor \frac{r_1-1}{r_2-1} + \frac{1}{r_2} \rfloor \leq \frac{r_1-1}{r_2-1}$ which implies $\lfloor \frac{r_1-1}{r_2-1} + \frac{1}{r_2} \rfloor \leq \lfloor \frac{r_1-1}{r_2-1} \rfloor$. Using (1) we obtain $N + 1 - \lfloor \frac{N+1}{r_2} \rfloor \geq r_1$. $\square$

4 Results for the general case

Using the results from Section 3, we derive bounds for general values of k_1. The main idea is to construct a copy of $KG^{r_1}(\lfloor \frac{n_1}{k_1} \rfloor, 1)$ in $KG^{r_1}(n_1, k_1)$ (see Theorem 3). Thus the existence of a homomorphism from $KG^{r_1}(n_1, k_1)$ to $KG^{r_2}(n_2, k_2)$ implies the existence of a homomorphism from $KG^{r_1}(\lfloor \frac{n_1}{k_1} \rfloor, 1)$ to $KG^{r_2}(n_2, k_2)$, which implies bounds on $\lfloor \frac{n_1}{k_1} \rfloor$ (see Corollary 1). On the other hand, homomorphisms from $KG^{r_1}(n_1, k_1)$ to $KG^{r_1}(n_1 - 2k_1 + 2, 1)$ are also shown to exist (see Theorem 3), which imply the existence of homomorphisms from $KG^{r_1}(n_1, k_1)$ to $KG^{r_2}(n_2, k_2)$ when homomorphisms from $KG^{r_1}(n_1 - 2k_1 + 2, 1)$ to $KG^{r_2}(n_2, k_2)$ do exist (see Corollary 1).

Theorem 3. *Let r, n, k be positive integers such that $n \geq rk$.*

(i) There exists a homomorphism $KG^r(m, 1) \to KG^r(n, k)$ if and only if $m \leq \lfloor \frac{n}{k} \rfloor$.

(ii) There exists a homomorphism $KG^r(n, k) \to KG^r(m, 1)$ if and only if $m \geq n - 2k + 2$.

Proof. We apply Theorem 1 and obtain that we can assume $r = 2$. Notice that $KG^2(m, 1) = K_m$ the complete graph with vertex set $[m]$. To prove (i), notice that $\vartheta(i) = \{(i-1)k+1, \ldots, ik\}$ defines a homomorphism $\vartheta : K_m \to KG^2(n, k)$ when $m \leq \lfloor \frac{n}{k} \rfloor$. On the other hand, if $\vartheta : K_m \to KG^2(n, k)$ is a homomorphism, then $\{\vartheta(i) : i \in [m]\}$ is a set of m pairwise disjoint subsets of $[n]$ of size k. Thus $mk \leq n$. To prove (ii), notice that any homomorphism from $KG^2(n, k)$ to K_m is a m-coloring of $KG^2(n, k)$ and thus the result follows from Lovász [4] result $\chi(KG^2(n, k)) = n - 2k + 2$. □

Now we use Theorem 3 and Theorem 2 to obtain necessary conditions for the existence of a homomorphism from $KG^{r_1}(n_1, k_1)$ to $KG^{r_2}(n_2, k_2)$.

Corollary 1. *Let $r_1, r_2, n_1, n_2, k_1, k_2$ be positive integers, with $n_i \geq r_i k_i$, for $i = 1, 2$, and with $r_1 > r_2 \geq 2$.*

(i) If there is a homomorphism from $KG^{r_1}(n_1, k_1)$ to $KG^{r_2}(n_2, k_2)$, then $\lfloor \frac{n_1}{k_1} \rfloor \leq \lfloor \frac{r_2(r_1-1)}{r_2-1} \rfloor$. In particular $n_1 < \frac{r_1 r_2 - 1}{r_2-1} k_1$.

(ii) If $n_1 - 2k_1 + 2 \leq \frac{r_2(r_1-1)}{r_2-1}$, then there exists a homomorphism from $KG^{r_1}(n_1, k_1)$ to $KG^{r_2}(n_2, k_2)$.

Proof. First, notice that (ii) follows from Theorem 3(ii) and Theorem 2. Now, we prove (i). Assume there is a homomorphism $KG^{r_1}(n_1, k_1) \to KG^{r_2}(n_2, k_2)$. Using Theorem 3, we have that there is a homomorphism from $KG^{r_1}(\lfloor \frac{n_1}{k_1} \rfloor, 1)$ to $KG^{r_1}(n_1, k_1)$. Therefore, by homomorphism composition, there is a homomorphism from $KG^{r_1}(\lfloor \frac{n_1}{k_1} \rfloor, 1)$ to $KG^{r_2}(n_2, k_2)$. Thus from Theorem 2 it follows that $\lfloor \frac{n_1}{k_1} \rfloor \leq \lfloor \frac{r_2(r_1-1)}{r_2-1} \rfloor$. Notice that this implies $\frac{n_1}{k_1} < \frac{r_2(r_1-1)}{r_2-1} + 1 = \frac{r_2 r_1 - 1}{r_2-1}$. □

Remark 2. *Notice that part (ii) of Corollary 1 gives sufficient conditions to the existence of special homomorphism between hypergraphs $KG^{r_1}(n_1, k_1)$ and $KG^{r_2}(n_2, k_2)$: the ones that map every hyperedge in $KG^{r_1}(n_1, k_1)$ to a single hyperedge in $KG^{r_2}(n_2, k_2)$. Thus the conditions in the corollary are tight for $k_1 = 1$. For $k_1 > 1$ Corollary 1 does not give a definite answer for $\frac{r_2(r_1-1)}{r_2-1} + 2k - 2 \leq n_1 \leq \frac{r_2(r_1-1)}{r_2-1} k + k$*

We end this section with the following example:

Example 2. *Let $n_1 = 8$, $k_1 = 2$, $r_1 = 4$, and $n_2 = 7$, $k_2 = 3$, $r_2 = 2$. As $n_1 - 2k_1 + 2 = 6 = \frac{r_2(r_1-1)}{r_2-1}$ then, by Corollary 1(ii), we know that there exists a homomorphism from $KG^4(8, 2)$ to $KG^2(7, 3)$. In fact, by Theorem 3(ii), there exists a homomorphism $\theta : KG^4(8, 2) \to KG^4(6, 1)$. It can be defined as follows: $\theta^{-1}(i) = \{\{i, j\} : i < j \leq 8\}$ for $1 \leq i \leq 5$, and $\theta^{-1}(6) = \{\{6, 7\}, \{6, 8\}, \{7, 8\}\}$. Now, let $n_1 = 6$, $k_1 = 1$, $r_1 = 4$, and $n_2 = 7$, $k_2 = 3$, $r_2 = 2$. By Theorem 2 ($(iii) \implies (i)$), there exists a homomorphism π from $KG^4(6, 1)$ to $KG^2(7, 3)$. Let $e = \{\{1, 2, 3\}, \{4, 5, 6\}\}$ be a hyperedge of $KG^2(7, 3)$. Now, define π as follows: $\pi^{-1}(\{1, 2, 3\}) = \{1, 2, 3\}$ and $\pi^{-1}(\{4, 5, 6\}) = \{4, 5, 6\}$. Notice that $\hat{\pi}([6]) = e$. Finally, the desired homomorphism ϕ from $KG^4(8, 2)$ to $KG^2(7, 3)$ can be defined by $\phi = \pi \circ \theta$. Moreover, as there exists a homomorphism $KG^4(8, 2) \to KG^2(7, 3)$, then $\lfloor \frac{8}{2} \rfloor = 4 \leq \lfloor \frac{2(4-1)}{2-1} \rfloor = 6$ as stated in Corollary 1(i).*

5 Relation to colorings

Some questions about colorings of hypergraphs can be reformulated as questions about hypergraph homomorphisms. Thus our results allow to characterize when certain types of colorings exist or not.

A *rainbow* t-coloring of a hypergraph $\mathcal{G}$ is a vertex coloring of $\mathcal{G}$ with t colors in which every hyperedge contains a vertex of each of the t colors. Notice that rainbow 2-coloring is the same as normal 2-coloring, and the existence of a rainbow t-coloring for $t = 2$ implies that the hypergraph is 2-colorable. Rainbow t-coloring is also known as *polychromatic* coloring where the basic question is: given a certain family of hypergraphs (often interpreted as set systems representing geometric objets), what is the smallest t that guarantees the existence of a rainbow t-coloring. We refer to the work of Bollobás et al. [2].

Notice that, for $r \geq 2$, the r-uniform Kneser hypergraph $KG^r(r, 1)$ is just a hyperedge with r vertices. Therefore, it is not difficult to see that a hypergraph $\mathcal{G}$ has a rainbow coloring with t colors if and only if there exists a homomorphism from $\mathcal{G}$ to $KG^t(t, 1)$. This notion leads us to characterize when an r-uniform Kneser hypergraph $KG^r(n, k)$ admits a rainbow t-coloring by using our results concerning the existence (or not) of a homomorphism from $KG^r(n, k)$ to $KG^t(t, 1)$.

On the other hand one can also be interested in colorings using exactly two colors per edge. A coloring with t colors using exactly two colors per edge is equivalent to a homomorphism to the complete graph K_t. Notice that $K_t = KG^2(t, 1)$ and thus our results allow to characterize when the hypergraph $KG^r(n, 1)$ admits such coloring, that is when $n < 2(r - 1)$, that is exactly when the graph is two colorable. In other words, any coloring of $KG^r(n, 1)$ with more than 2 colors necessarily colors one of the edges of $KG^r(n, 1)$ with 3 or more colors. Similar results can be obtained for other uniform Kneser hypergraphs.

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