On the P_3-hull number of Hamming graphs

Boštjan Brešar[a,b] Mario Valencia-Pabon[c]

October 22, 2019

[a] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

[b] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

[c] Université Paris-13, Sorbonne Paris Cité LIPN, CNRS UMR7030, Villataneuse, France

Abstract

Let S be a subset of vertices of a finite and simple graph G. If every vertex having two neighbors inside S is also in S, then S is P_3-convex. The P_3-convex hull $H(S)$ of S is the smallest P_3-convex set containing S. If $H(S) = V(G)$, then S is a P_3-hull set of G. The cardinality $h(G)$ of a minimum P_3-hull set in G is called the P_3-hull number of G. In this paper, we prove that the P_3-hull number of a Hamming graph $K_{r_1}\square\cdots\square K_{r_n}$ which is the n-fold Cartesian product of complete graphs K_{r_i}, with $r_i > 1$ for $1 \le i \le n$, is equal to $\lceil n/2\rceil + 1$.

Keywords: P_3-convexity, P_3-hull number, Cartesian product, Hamming graphs

AMS Subj. Class. (2010): 05C76, 52A37, 05C85

1 Introduction

We consider finite, simple, and undirected graphs. For a finite and simple graph $G = (V, E)$, a *graph convexity* on V is a collection $\mathcal{C}$ of subsets of V such that $\emptyset, V \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections. The sets in $\mathcal{C}$ are called *convex sets* and the *convex hull* $H_{\mathcal{C}}(S)$ in $\mathcal{C}$ of a set S of vertices of G is the smallest set in $\mathcal{C}$ containing S (see the monograph [25] for a comprehensive survey on convex structures). Some natural convexities in graphs are defined by a set $\mathcal{P}$ of paths in G, in a way that a set S of vertices of G is convex if and only if for every path $P : v_0, v_1, \ldots, v_l \in \mathcal{P}$ such that v_0 and v_l belong to S, all vertices of P belong to S, cf. [7]. If we define $\mathcal{P}$ as the set of all shortest paths in G, we have the well-known *geodetic convexity* (see [6, 12, 13, 21] and references therein). The *monophonic convexity* is defined by considering $\mathcal{P}$ as the set of all induced paths of G [14, 15, 17].

If we let $\mathcal{P}$ be the set of all paths of G with three vertices, we have the well-known P_3-*convexity* which will be studied in this paper. The P_3-convexity has been studied in directed graphs [16, 20, 24] and undirected graphs [4, 8, 9, 10, 11, 22]. Given a set $S \subseteq V$, the P_3-*interval* $I[S]$ of S is formed by S, together with every vertex outside S with at least two neighbors in S. If $I[S] = S$, then the set S is P_3*-convex*. The P_3*-convex hull* $H_{\mathcal{C}}(S)$ of S is the smallest P_3-convex set containing S. In what follows, we write $H(S)$ instead of $H_{\mathcal{C}}(S)$. The P_3-convex hull $H(S)$ can be formed from the sequence $I^p[S]$, where p is a nonnegative integer, $I^0[S] = S$, $I^1[S] = I[S]$, and $I^p[S] = I[I^{p-1}[S]]$, for every $p \geq 2$. When for some $p \in \mathbb{N}$, we have $I^q[S] = I^p[S]$, for all $q \geq p$, then $I^p[S]$ is a P_3-convex set. If $H(S) = V(G)$ we say that S is a P_3*-hull set* of G. The cardinality $h(G)$ of a minimum P_3-hull set in G is called the P_3*-hull number* of G. Concerning the P_3-hull number, some results are described in [8, 10, 11], and even earlier this concept was considered in the context of spreading disease on the square grid [2, 5]. Recently, it has been shown that computing the P_3-hull number of a graph is an NP-hard problem even in the case of the Cartesian product $G \Box K_2$ of an arbitrary connected graph G with the complete graph on two vertices [11].

In this paper, we consider the P_3-hull number in the class of Hamming graphs, which are the n-fold Cartesian products of arbitrary complete graphs. Hamming graphs enjoy several nice propertices and present a natural model for a communication network. In particular, vertices of a Hamming graph can be labeled in such a way that the distance between two vertices coincides with the Hamming distance of their labels; see [18]. In addition, Hamming graphs can be recognized in linear time and space [19]. A special and important subclass of Hamming graphs are (ubiquitous) hypercubes, obtained when all factors are complete graphs of order 2. Different types of convexities and closures were considered in relation with Hamming graphs [1, 3, 23]. In this paper, we contrast the result of Centeno et al. [11] by proving that the P_3-hull number of a Hamming graph can be computed efficiently. More precisely, our main result states that the P_3-hull number of a Hamming graph $K_{r_1} \Box \cdots \Box K_{r_n}$, where $r_i > 1$ for $1 \leq i \leq n$, is equal to $\lceil n/2 \rceil + 1$.

This paper is organized as follows: In Section 2 we introduce the notation and terminology. Section 3 describes the structure of P_3-convex sets in general Cartesian products of graphs. Section 4 presents results concerning the P_3-convexity of Hamming graphs. Finally, we give some concluding remarks in Section 5.

2 Preliminaries

Let $G = (V, E)$ be graph and let $S \subseteq V$. We denote by $<S>$ the subgraph of G induced by the vertices in S, that is, $<S> = (S, E_S)$, where $E_S = \{\{x, y\} : x, y \in S \text{ and } \{x, y\} \in E\}$. The *distance* in a graph G between two vertices x and y is defined as the length of a shortest path between x and y, and we denote it by $d_G(x, y)$. The *diameter* of G, diam(G), is the maximum distance between any two vertices in G. The *distance* $d_G(x, S)$ between a vertex $x \in V(G)$ and a set $S \subset V(G)$ is defined as $\min\{d_G(x, y) : y \in S\}$, and the distance $d_G(S, S')$ between two subsets S, S' of $V(G)$ is defined as $\min\{d_G(x, y) : x \in S, y \in S'\}$.

Let $H = (V_H, E_H)$ and $G = (V_G, E_G)$ be two graphs. A *homomorphism* from H to G is a mapping from V_H to V_G which preserves adjacencies (i.e;, edges are mapped to edges). Two graphs H and G are said *isomorphic* if there exist injective homomorphisms from H to G and from G to H.

Let G_i, $i \in [n]$, be arbitrary graphs. (We let $[n] = \{1 \ldots, n\}$.) The *Cartesian product* $\Box_{i \in [n]} G_i$ (written also as $G_1 \Box \cdots \Box G_n$) has $V(G_1) \times \cdots \times V(G_n)$ as the vertex set, and two vertices $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ are adjacent if there exists an index $j\in[n]$ such that $x_jy_j\in E(G_j)$ and $x_i=y_i$ for all $i\neq j$. The Cartesian product $\Box_{i\in[n]}K_2$ is called *n-cube* or *hypercube* of dimension n. More generally, if for each $i\in[n]$ the graph G_i is a complete graph K_{r_i}, then $\Box_{i\in[n]}G_i$ is a *Hamming graph*. (Note that the complete graphs K_{r_i} may be of arbitrary orders.)

For an arbitrary vertex $x=(x_1,\dots,x_n)\in V(\Box_{i\in[n]}G_i)$ the *j-fiber* of $\Box_{i\in[n]}G_i$ *at vertex* x is the subset of $V(\Box_{i\in[n]}G_i)$ induced by $\{x_1\}\times\cdots\times\{x_{j-1}\}\times V(G_j)\times\{x_{j+1}\}\times\cdots\times\{x_n\}$. If $\Box_{i\in[n]}G_i$ is denoted in short by X, we denote by X_j^x the j-fiber of X at vertex x. Clearly, $<X_j^x>$ is isomorphic to G_j for every $x\in V(X)$ and every $j\in[n]$. It is also obvious that $\cup_{x\in V(X)}X_j^x=V(X)$ for any $j\in[n]$. If $S\subset V(X)$, then $p_i(S)$ denotes the projection of S to the ith coordinate, that is, $p_i(S)=\{x_i\in V(G_i)\,:\,(x_1,\cdots,x_i,\cdots,x_n)\in S\}$.

3 P_3-convex sets in Cartesian products

The following result about P_3-convex sets in general Cartesian products of graphs will be used in Section 4 for establishing the hull number of Hamming graphs.

Theorem 1. *Let $X=\Box_{i\in[n]}G_i$ be the Cartesian product of graphs $G_1,\dots,G_n$, $n\geq 1$. If S is a P_3-convex set of X such that the subgraph $<S>$ induced by S is connected, then $S=Z_1\times\cdots\times Z_n$, where each Z_i is a P_3-convex set of the graph G_i.*

Proof. Let $i\in[n]$ be an arbitrarily chosen index. We first claim that there exists $Z_i\subset V(G_i)$ such that for every $x\in V(X)$ we either have $S\cap X_i^x=\emptyset$ or $p_i(S\cap X_i^x)=Z_i$. Let $x\in V(X)$ such that $S\cap X_i^x\neq\emptyset$ (since S is non-empty, such x exists). Let $x=(x_1,\dots,x_n)$. If $S=S\cap X_i^x$, then we are done, because for any y such that $X_i^y\neq X_i^x$ we have $S\cap X_i^y=\emptyset$, hence we may write $Z_i=p_i(S\cap X_i^x)$. Therefore, we may assume that $S\cap X_i^x\subsetneq S$, and since S is connected, there exists a vertex $y\in S$ such that $xy\in E(G)$ and $y\notin X_i^x$. Let $j\in[n]$ be the index such that $x_j\neq y_j$. Note that $j\neq i$, and assume without loss of generality that $i<j$. We claim that $X_i^x\cap S=X_i^y\cap S$. Suppose this is not the case. Then we derive that the following situation appears: $x=(x_1\dots,x_n)\in S$, $z=(x_1,\dots,x_{i-1},z_i,x_{i+1},\dots,x_n)\in S$ where $x_iz_i\in E(G_i)$, $y=(x_1,\dots,x_{j-1},y_j,x_{j+1},\dots,x_n)\in S$, however vertex $(x_1,\dots,x_{i-1},z_i,x_{i+1},\dots,x_{j-1},y_j,x_{j+1},\dots,x_n)$ is not in S. But the latter vertex is a common neighbor of vertices y and z, which are both in S. This is a contradiction with S being P_3-convex, and we conclude, by using also connectedness of $<S>$ and induction, that there exists a set $Z_i\subseteq V(G_i)$ such that $p_i(S\cap X_i^x)=Z_i$ for all $x\in X$ with $S\cap X_i^x\neq\emptyset$.

Since $i\in[n]$ was arbitrarily chosen, we infer that S is equal to $Z_1\times\cdots\times Z_n$. Suppose that Z_i is not a P_3-convex set in G_i for some $i\in[n]$. Hence there exists a vertex $u\in V(G_i)-Z_i$ that has two neighbors a and b in Z_i. Now, let $x\in V(X)$ such that $S\cap X_i^x\neq\emptyset$. Since $p_i(S\cap X_i^x)=Z_i$, we infer that $(x_1,\dots,x_{i-1},u,x_{i+1},\dots,x_n)$ is not in S, but two of its neighbors $(x_1,\dots,x_{i-1},a,x_{i+1},\dots,x_n)$ and $(x_1,\dots,x_{i-1},b,x_{i+1},\dots,x_n)$ belong to S. This is a contradiction with S being P_3-convex, therefore each Z_i is P_3-convex in G_i. $\Box$

4 P_3-convexity in Hamming graphs

Recall that Hamming graph $\mathcal{H}$ is a graph isomorphic to an n-fold Cartesian product of arbitrary complete graphs, that is, $\mathcal{H} = K_{r_1} \Box K_{r_2} \Box \ldots \Box K_{r_n}$, with $n > 1$ and $r_i > 0$ for $1 \leq i \leq n$.

Let $\mathcal{H} = K_{r_1} \Box K_{r_2} \Box \ldots \Box K_{r_n}$ be a Hamming graph, with $n > 1$. Note that K_1 is a unit of the Cartesian product, which is also associative and commutative. We may thus assume that $r_i > 1$ for $1 \leq i \leq n$ (since $K_1 \Box \cdots \Box K_1 \Box K_{r_k} \Box K_{r_{k+1}} \Box \cdots \Box K_{r_n}$ is isomorphic to $K_{r_k} \Box K_{r_{k+1}} \Box \ldots \Box K_{r_n}$). Under this assumption, we say that n is the *dimension* of $\mathcal{H}$ (and if exactly k of the factors are isomorphic to K_1, then the dimension is $n - k$).

From Theorem 1 we infer the following consequence about P_3-convex sets in Hamming graphs. (The last statement of the corollary follows from the fact that the P_3-convex hull of a pair of vertices in a clique equals the entire clique.)

Corollary 2. *Let $\mathcal{H} = K_{r_1} \Box K_{r_2} \Box \ldots \Box K_{r_n}$ be a Hamming graph, with $n > 1$ and $r_i > 1$ for $1 \leq i \leq n$. If S is a P_3-convex set of $\mathcal{H}$ such that the subgraph $<S>$ induced by S is connected, then $<S>$ is isomorphic to a Hamming graph with dimension k, for some $1 \leq k \leq n$. In particular, $<S>$ is isomorphic to $K_{\ell_1} \Box K_{\ell_2} \Box \ldots \Box K_{\ell_n}$, where $\ell_i = r_i$ holds for k of the indices, and $\ell_i = 1$ for the other $n - k$ indices.*

Next, we prove another property about P_3-convex sets in Hamming graphs that gives an idea how one can build a P_3-convex hull in Hamming graphs, which will be used later.

Lemma 3. *Let $\mathcal{H} = K_{r_1} \Box K_{r_2} \Box \ldots \Box K_{r_n}$ be a Hamming graph, with $n > 1$ and $r_i > 1$ for $1 \leq i \leq n$, and let $S \subset V(\mathcal{H})$ be a P_3-convex set in $\mathcal{H}$ such that $<S>$ is a connected subgraph; i.e., $<S>$ is a Hamming graph and let k be its dimension.*

(i) If $x \in V(\mathcal{H})$ such that $d_{\mathcal{H}}(x, S) = 1$, then $H(S \cup \{x\})$ induces a Hamming (sub)graph of dimension $k + 1$.

(ii) If $x \in V(\mathcal{H})$ such that $d_{\mathcal{H}}(x, S) = 2$, then $H(S \cup \{x\})$ induces a Hamming (sub)graph of dimension $k + 2$.

(iii) If $x \in V(\mathcal{H})$ such that $d_{\mathcal{H}}(x, S) > 2$, then $H(S \cup \{x\})$ is the disjoint union of $<S>$ and $<\{x\}>$.

Proof. By the assumptions, there is a vertex x at distance at least 1 from S, which implies that S is a proper subset of $V(\mathcal{H})$. If $|S| = 1$, then the statement of the lemma is clear, hence we may assume that $|S| \geq 2$.

Consider first the case (i), when $d_{\mathcal{H}}(x, S) = 1$, and let $y \in S$ be the vertex such that $d_{\mathcal{H}}(x, y) = 1$. Since $|S| \geq 2$, there is at least one neighbor $z \in S$ of y. It is clear that $d_{\mathcal{H}}(x, z) = 2$, and the other common neighbor x' of x and z also belongs to $H(S \cup \{x\})$, by definition of P_3-convex hull. Note that $d_{\mathcal{H}}(x', S) = 1$. Let w be a neighbor of z in S, different from y. Then, there is a unique common neighbor of x' and w, which we denote by x''. As above, note that $d_{\mathcal{H}}(x'', S) = 1$. Continuing with this process, and using the connectedness of $<S>$, we infer that $H(S \cup \{x\})$ contains a subset of vertices in $\mathcal{H}$ which induces a subgraph isomorphic to $<S> \Box K_2$. To complete the argument, we may assume without loss of generality that $<S>$ is isomorphic to $K_{r_1} \Box \cdots \Box K_{r_k} \Box K_1 \Box \cdots \Box K_1$, and let $k+1$ be the coordinate in which x and y differ. If $r_{k+1} = 2$, then we are done, since $H(S \cup \{x\})$ is then isomorphic to $K_{r_1} \Box \cdots \Box K_{r_k} \Box K_2 \Box K_1 \Box \cdots \Box K_1$, which is P_3-convex in $\mathcal{H}$, and so it is a Hamming (sub)graph of dimension $k+1$. Otherwise, x and y have $r_{k+1}-2$ other common neighbors, all of which differ in the $(k+1)$-st coordinate, and they all belong to $H(S \cup \{x\})$. By applying the same procedure as above, using the connectedness of $<S>$, we infer that $H(S \cup \{x\})$ contains a subset of $\mathcal{H}$ isomorphic to $<S> \Box K_{r_{k+1}}$, which is P_3-convex in $\mathcal{H}$. This is the Hamming subgraph of $\mathcal{H}$, isomorphic to $K_{r_1} \Box \cdots \Box K_{r_k} \Box K_{r_{k+1}} \Box K_1 \Box \cdots \Box K_1$.

The case (ii), when $d_{\mathcal{H}}(x, S) = 2$ can be proved by using the previous case. First note that if $y \in S$ is the vertex such that $d_{\mathcal{H}}(x, y) = 2$ and $x' \notin S$ is a common neighbor of x and y, then we arrive at the previous case: a vertex $x' \in H(S \cup \{x\})$ is at distance 1 from $y \in S$. As in the previous case, we obtain that $H(S \cup \{x'\})$ induces a Hamming subgraph R of dimension $k+1$. Now, x is not in R, and $d_{\mathcal{H}}(x, R) = 1$. Hence we can again apply the previous case, and derive that $<H(S \cup \{x\})>$ is a Hamming subgraph of $\mathcal{H}$ of dimension $k+2$. The last case, (iii), is obvious. $\Box$

By using analogous arguments as in Lemma 3 one can derive the following result.

Lemma 4. *Let $\mathcal{H} = K_{r_1} \Box K_{r_2} \Box \ldots \Box K_{r_n}$ be a Hamming graph, with $n > 1$ and $r_i > 1$ for $1 \leq i \leq n$, and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hamming subgraphs of $\mathcal{H}$ with dimension k and k', respectively.*

- *(i) If $d_{\mathcal{H}}(V(\mathcal{H}_1), V(\mathcal{H}_2)) = 1$, then $H(V(\mathcal{H}_1) \cup V(\mathcal{H}_2))$ induces a Hamming (sub)graph of dimension at most $k + k' + 1$.*
- *(ii) If $d_{\mathcal{H}}(V(\mathcal{H}_1), V(\mathcal{H}_2)) = 2$, then $H(V(\mathcal{H}_1) \cup V(\mathcal{H}_2))$ induces a Hamming (sub)graph of dimension at most $k + k' + 2$.*
- *(iii) If $d_{\mathcal{H}}(V(\mathcal{H}_1), V(\mathcal{H}_2)) > 2$, then $H(V(\mathcal{H}_1) \cup V(\mathcal{H}_2))$ is the disjoint union of $V(\mathcal{H}_1)$ and $V(\mathcal{H}_2)$.*

Lemma 5. *Let $\mathcal{H} = K_{r_1} \Box K_{r_2} \Box \ldots \Box K_{r_n}$ be a Hamming graph, with $n > 1$ and $r_i > 1$ for $1 \leq i \leq n$, and let $P \subset V(\mathcal{H})$ such that $<H(P)>$ is a connected subgraph of $\mathcal{H}$. If $|P| = k$, where $2k - 2 \leq n$, then $H(P)$ induces a Hamming (sub)graph of dimension at most $2k - 2$.*

Proof. The proof is by induction on the cardinality k of the set P. If $k = 1$, then $H(P)$ is a single vertex, inducing the Hamming graph K_1, which is of dimension $0 = 2k - 2$, hence the statement is true. For the induction step, consider a set P in a Hamming graph $\mathcal{H}$ of dimension n, such that $|P| = k$, $<H(P)>$ is connected, and $2k - 2 \leq n$.

Suppose that there exists a vertex $x \in P$, such that $H(P - \{x\})$ induces a connected subgraph. Since $|P - \{x\}| = k - 1$, we may use the induction hypothesis to derive that $<H(P - \{x\})>$ is a Hamming subgraph of $\mathcal{H}$ of dimension at most $2(k-1) - 2 = 2k - 4$. Since $H(P)$ induces a connected graph, we infer that $d_{\mathcal{H}}(x, P - \{x\}) \leq 2$. Hence, by using Lemma 3, the P_3-convex hull $H(P)$ induces a Hamming (sub)graph of dimension at most $2k - 2$.

Now, let R be a smallest subset of P such that $<H(P-R)>$ is a connected subgraph of $\mathcal{H}$. Let $|R| = r$. By the minimality of r, $<H(R)>$ is a connected subgraph of $\mathcal{H}$. By using induction hypothesis, we infer that $<H(R)>$ is isomorphic to a Hamming graph of dimension at most $2r-2$. From the same reason, the dimension of $<H(P-R)>$ is at most $2(k-r)-2$. Now, since $<H(P)>$ is a connected subgraph of $\mathcal{H}$, we have $d_{\mathcal{H}}(R, P-R) \leq 2$ (using Lemma 4(iii)). By Lemma 4, $H(R \cup (P-R))$ is a Hamming (sub)graph of dimension at most $H(R) + H(P-R) + 2 = 2r-2+2(k-r)-2+2 = 2k-2$. Since, $R \cup (P-R) = P$, this means that $H(P)$ induces a Hamming graph of dimension at most $2k-2$. $\square$

Corollary 6. *If $\mathcal{H} = K_{r_1} \square K_{r_2} \square \ldots \square K_{r_n}$ is a Hamming graph, with $n > 1$ and $r_i > 1$ for $1 \leq i \leq n$, then $h(\mathcal{H}) \geq \lceil \frac{n}{2} \rceil + 1$.*

Proof. Let P be a set of vertices in $\mathcal{H}$ such that $<H(P)>$ is a connected subgraph of $\mathcal{H}$, and let $|P| = k$, where $2k-2 \leq n$. By Lemma 5, $H(P)$ induces a Hamming subgraph of dimension at most $2k-2$. Now, if $2k-2 < n$, then P is not a P_3-hull set of $\mathcal{H}$. This implies that a P_3-hull set of $\mathcal{H}$ must be of cardinality at least $(n+2)/2$ for even n, and $(n+3)/2$ for odd n, respectively. Thus, $h(\mathcal{H}) \geq \lceil \frac{n}{2} \rceil + 1$. $\square$

To prove the reversed inequality of Corollary 6, we first focus on the most prominent class of Hamming graphs—the hypercubes.

For any $n \in \mathbb{Z}^+$, the *n-dimensional hypercube* (or *n-cube*), denoted Q_n, is the graph in which the vertices are all binary n-tuples of length n (i.e., the set $\{0,1\}^n$), and two vertices are adjacent if and only if they differ in exactly one position. It is well known and easy to see that the distance $d_{Q_n}(x, y)$ between two vertices x and y of the n-cube is equal to the number of positions in which these two vertices differ.

Clearly, for any $n \in \mathbb{N}$ the hypercube Q_n is isomorphic to the graph $Q_{n-1} \square K_2$. Therefore, Q_n can be recursively constructed from n copies of the graph K_2, inferring $Q_n = \underbrace{K_2 \square \cdots \square K_2}_{n \text{ times}}$, which demonstrates their position in the class of Hamming graphs.

Lemma 7. *Let n be a positive integer. Then, $h(Q_n) \leq \lceil \frac{n}{2} \rceil + 1$.*

Proof. Defining the set S as follows:

$$S = \{00\ldots0, 0\ldots011, 0\ldots01111, \ldots\ldots, 01\ldots1, 1\ldots1\}$$

if n is odd, and

$$S = \{00\ldots0, 0\ldots011, 0\ldots01111, \ldots\ldots, 001\ldots1, 1\ldots1\}$$

if n is even, gives, by using Lemma 3, the P_3-hull set of Q_n. In fact, notice that the set of vertices $H(\{00\ldots0, 0\ldots011\})$ induces a (sub)hypercube of dimension 2 which is connected and isomorphic to the cycle on four vertices. Therefore, we can use several times Lemma 3(ii) (and one time Lemma 3(i), when n is odd) in order to obtain that S is a P_3-hull set of Q_n. Clearly, in the first case, $|S| = \frac{n+1}{2} + 1$, and in the second case, $|S| = \frac{n}{2} + 1$, which gives the claimed upper bound. $\square$

In a similar way, by also using Lemma 7, we get the result for an arbitrary Hamming graph as follows.

Lemma 8. *Let $\mathcal{H} = K_{r_1} \square K_{r_2} \square \ldots \square K_{r_n}$ be a Hamming graph, with $n > 1$ and $r_i > 1$ for $1 \leq i \leq n$. Then, $h(\mathcal{H}) \leq \lceil \frac{n}{2} \rceil + 1$.*

Proof. Let S be the set defined as in the proof of Lemma 7 and recall that $[n] = \{1, 2, \ldots, n\}$. Then, $|S| = \lceil \frac{n}{2} \rceil + 1$ and note that the hypercube $Q_n = K_2 \square K_2 \square \ldots \square K_2$ is an induced subgraph of the graph $<H(S)>$. Let i be any element in $[n]$ such that $|V(K_{r_i})| \geq 3$ and let A_i be the n-fold Cartesian product $\{0,1\} \times \ldots \times \{2, 3, \ldots, r_i - 1\} \times \ldots \{0, 1\}$. Notice that the subgraph $G_i = K_2 \square \ldots \square K_2 \square K_{r_i} \square K_2 \square \ldots \square K_2$ is an induced subgraph of $<H(S)>$. In fact, any vertex in A_i has two neighbors in the subgraph isomorphic to Q_n. Therefore, for all $i \in [n]$, the subgraph G_i induced by the vertices in A_i is an induced subgraph of $<H(S)>$. Now, let $k > 1$ and let $\{i_1, \ldots, i_k\}$ be any k-set in $[n]$, such that $|V(K_{r_{i_t}})| \geq 3$, for $1 \leq t \leq k$. Assume that for all $(k-1)$-sets $\{j_1, \ldots, j_{k-1}\}$ in $[n]$ such that $|V(K_{r_{j_s}})| \geq 3$, for $1 \leq s \leq k-1$, the subgraphs formed by the n-fold Cartesian product of the $k-1$ complete graphs $K_{r_{j_s}}$ and the $n-k+1$ graphs K_2 are induced subgraphs of $<H(S)>$. Clearly, the graph induced by n-fold Cartesian product of the k complete graphs K_{i_t}, with $1 \leq t \leq k$, and the $n-k$ other graphs K_2, is an induced subgraph of $<H(S)>$, because each vertex of the former one has at least two neighbors in $<H(S)>$. Therefore, by induction, the result follows. $\square$

By Lemma 8 and Corollary 6, we have the following result, which also implies that the time complexity of computing the P_3-hull number of Hamming graphs is constant.

Theorem 9. *Let $\mathcal{H} = K_{r_1} \square K_{r_2} \square \ldots \square K_{r_n}$ be a Hamming graph, with $n > 1$ and $r_i > 1$ for $1 \leq i \leq n$. Then, $h(\mathcal{H}) = \lceil \frac{n}{2} \rceil + 1$. In particular, $h(Q_n) = \lceil \frac{n}{2} \rceil + 1$.*

5 Concluding remarks

Coelho et al. [11] give the following results concerning the P_3-hull number of graphs.

Definition 1 (Definition 3.10 in [11])**.** *Let G be a connected graph. The graph G is of Type 1, if there exists a minimum P_3-hull set $S \subseteq V(G)$ that can be partitioned in two nonempty disjoint sets A and B, with $S = A \cup B$, in which $d(H(A), H(B)) \leq 1$.*

Theorem 10 (Theorem 3.13 in [11])**.** *Let G be a nontrivial connected graph and let $n \geq 2$ be an integer. Then,*

$$h(G \square K_n) = \begin{cases} h(G), & \text{if } G \text{ is of Type 1;} \\ h(G) + 1, & \text{otherwise.} \end{cases}$$

Unfortunately, determining if a graph G is of Type 1 is a hard problem as has been shown in [11]:

Corollary 11 (Corollary 3.19 in [11])**.** *Deciding whether a graph G is of Type 1 is NP-complete.*

However, for the family of Hamming graphs, by Theorem 9 and Theorem 10, we can deduce directly the following result.

Corollary 12. *Let $\mathcal{H}$ be the Hamming graph $K_{r_1}\square\cdots\square K_{r_n}$ defined by the n-fold Cartesian product of complete graphs K_{r_i}, with $n > 1$ and $r_i > 1$ for $1 \leq i \leq n$. Then, $\mathcal{H}$ is of Type 1 if and only if n is even.*

Acknowledgements

B. B. acknowledges the financial support from the Slovenian Research Agency (project grant No. J1-9109 and research core funding No. P1-0297). M. V-P. acknowledges the financial support from the Paris-Nord University, the Franco-Belgian "Tournesol" PHC project grant No. 42703TD and the LIA INFINIS/SINFIN (CNRS France - CONICET Argentina).

References

[1] D. Avis, H. Maehara, Metric extensions and the L^1 hierarchy, Discrete Math. 131 (1994) 17-28.

[2] J. Balogh, G. Pete, Random disease on the square grid, Random Structures Algorithms 13 (1998) 409-422.

[3] H.-J. Bandelt, A. Röhl, Quasi-median hulls in Hamming space are Steiner hulls, Discrete Appl. Math. 157 (2009) 227-233.

[4] R. M. Barbosa, E. M. M. Coelho, M. C. Dourado, D. Rautenbach, J. L. Szwarcfiter, On the Carathéodory number for the convexity of paths of order three, SIAM J. Discrete Math. 26 (2012) 929-939.

[5] B. Bollobás, *The Art of Mathematics: Coffee Time in Memphis*, Cambridge University Press, 2006.

[6] B. Brešar, M. Kovše, A. Tepeh, Geodetic sets in graphs, in: M. Dehmer ed., *Structural Analysis of Complex Networks*, Birkhäuser, Boston, (2010) 197–218.

[7] J. R. Calder, Some elementary properties of interval convexities, J. London Math. Soc. 3 (1971) 422–428.

[8] V. Campos, R. M. Sampaio, A. Silva, J. L. Szwarcfiter, Graphs with few P_4's under the convexity of paths of order three, Discrete Appl. Math. 192 (2015) 28-39.

[9] C. C. Centeno, S. Dantas, M. C. Dourado, D. Rautenbach, J. L. Szwarcfiter, Convex partitions of graphs induced by paths of order three, Discrete Math. Theor. Comput. Sci. 12 (2010) 175-184.

[10] C. C. Centeno, L. D. Penso, D. Rautenbach, V. G. P. de Sá, Geodetic number versus hull number in P_3-convexity, SIAM J. Discrete Math. 27 (2013) 717-731.

[11] E. M. M. Coelho, H. Coelho, J. R. Nascimento, J. L. Szwarcfiter, On the P_3-hull number of some products of graphs, Discrete Appl. Math. 253 (2019) 2-13.

[12] M. C. Dourado, L. D. Penso, D. Rautenbach, On the geodetic hull number of P_k-free graphs, Theoret. Comput. Sci. 640 (2016) 52-60.

[13] M. C. Dourado, F. Protti, D. Rautenbach, J. L. Szwarcfiter, On the hull number of triangle-free graphs, SIAM J. Discrete Math. 23 (2010) 2163-2172.

[14] M. C. Dourado, F. Protti, J. L. Szwarcfiter, Complexity results related to monophonic convexity, Discrete Appl. Math. 158 (2010) 1268-1274.

[15] P. Duchet, Convex sets in graphs, II Minimal path convexity, J. Combin. Theory Ser. B 44 (1988) 307-316.

[16] P. Erdös, E. Fried, A. Hajnal, E. Milner, Some remarks on simple tournaments, Algebra Universalis 2 (1972) 238-245.

[17] M. Farber, R. E. Jamison, Convexity in graphs and Hypergraphs, SIAM J. Algebraic Discrete Methods 7 (1986) 433-444.

[18] R. Hammack, W. Imrich, S. Klavžar, *Handbook of product graphs*, CRC press, Boca Raton, FL, 2011.

[19] W. Imrich, S. Klavžar, Recognized Hamming graphs in linear time and space, Inform. Process. Lett. 2 (1997) 91-95.

[20] D. B. Parker, R. F. Westhoff, M. J. Wolf, On two-path convexity in multipartite tournaments, European J. Combin. 29 (2008) 641-651.

[21] I. M. Pelayo, *Geodesic Convexity in Graphs*, Springer, New York, 2013.

[22] L. D. Penso, F. Protti, D. Rautenbach, U. dos Santos Souza, Complexity analysis of P_3-convexity problems on bounded-degree and planar graphs, Theoret. Comput. Sci. 607 (2015) 83-95.

[23] V. Soltan, Metric convexity in graphs, Studia Univ. Babeş-Bolyai Math. 36 (1991) 3-43.

[24] J. C. Varlet, Convexity in tournaments, Bull. Soc. Roy. Sci. Liège 45 (1976) 570-586.

[25] M. L. J. van de Vel, *Theory of Convex Structures*, North Holland, Amsterdam, 1993.