k-tuple colorings of the cartesian product of graphs*

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Abstract

A k-tuple coloring of a graph G assigns a set of k colors to each vertex of G such that if two vertices are adjacent, the corresponding sets of colors are disjoint. The k-tuple chromatic number of G, $\chi_k(G)$, is the smallest t so that there is a k-tuple coloring of G using t colors. It is well known that $\chi(G\square H) = \max\{\chi(G), \chi(H)\}$. In this paper, we show that there exist graphs G and H such that $\chi_k(G\square H) > \max\{\chi_k(G), \chi_k(H)\}$ for $k \geq 2$. Moreover, we also show that there exist graph families such that, for any $k \geq 1$, the k-tuple chromatic number of their cartesian product is equal to the maximum k-tuple chromatic number of its factors.

keyword: k-tuple colorings, Cartesian product of graphs, Kneser graphs, Cayley graphs, Hom-idempotent graphs.

1 Introduction

A classic *coloring* of a graph G is an assignment of colors (or natural numbers) to the vertices of G such that any two adjacent vertices are assigned different colors. The smallest number t such that G admits a coloring with t colors (a *t-coloring*) is called the *chromatic number* of G and is denoted by $\chi(G)$. Several generalizations of the coloring problem have been introduced in the literature, in particular, cases in which each vertex is assigned not only a color but a set of colors, under different restrictions. One of these variations is the *k-tuple coloring* introduced independently by Stahl [11] and Bollobás and Thomason [3]. A k-tuple coloring of a graph G is an assignment of k colors to each vertex in such a way that adjacent vertices are assigned distinct colors. The *k-tuple coloring problem* consists into finding the minimum number of colors in a k-tuple coloring of a graph G, which we denote by $\chi_k(G)$.

The *cartesian product* $G\square H$ of two graphs G and H has vertex set $V(G) \times V(H)$, two vertices being joined by an edge whenever they have one coordinate equal and the other adjacent. This product is commutative and associative up to isomorphism. There is a simple formula expressing the chromatic number of a cartesian product in terms of its factors:

$$\chi(G\square H) = \max\{\chi(G), \chi(H)\}. \qquad (1)$$

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The identity (1) admits a simple proof first given by Sabidussi [10].

The *Kneser graph* $K(m,n)$ has as vertices all n-element subsets of the set $[m] = \{1,\ldots,m\}$ and an edge between two subsets if and only if they are disjoint. We will assume in the rest of this work that $m \geq 2n$, otherwise $K(m,n)$ has no edges. The Kneser graph $K(5,2)$ is the well known *Petersen Graph*. Lovász [9] showed that $\chi(K(m,n)) = m-2n+2$. The value of the k-tuple chromatic number of the Kneser graph is the subject of an almost 40-year-old conjecture of Stahl [11] which asserts that: if $k = qn-r$ where $q \geq 0$ and $0 \leq r < n$, then $\chi_k(K(m,n)) = qm-2r$. Stahl's conjecture has been confirmed for some values of k, n and m [11, 12].

An *homomorphism* from a graph G into a graph H, denoted by $G \to H$, is an edge-preserving map from $V(G)$ to $V(H)$. It is well known that an ordinary graph coloring of a graph G with m colors is an homomorphism from G into the complete graph K_m. Similarly, an n-tuple coloring of a graph G with m colors is an homomorphism from G into the Kneser graph $K(m,n)$. A graph G is said *hom-idempotent* if there is an homomorphism $G\Box G \to G$. We denote by $G \not\to H$ if there exists no homomorphism from G to H. The *clique number* of a graph G, denoted by $\omega(G)$, is the maximum size of a *clique* in G (i.e., a complete subgraph of G). Clearly, for any graphs G and H, we have that $\chi(G) \geq \omega(G)$ (and so, $\chi_k(G) \geq \chi_k(K_{\omega(G)}) = k\omega(G)$) and, if there is an homomorphism from G to H then, $\chi(G) \leq \chi(H)$ and, moreover, $\chi_k(G) \leq \chi_k(H)$.

In this paper, we show that equality (1) does not hold in general for k-tuple colorings of graphs. In fact, we show that for some values of $k \geq 2$, there are Kneser graphs $K(m,n)$ for which $\chi_k(K(m,n)\Box K(m,n)) > \chi_k(K(m,n))$. Surprisingly, there exist some Kneser graphs $K(m,n)$ for which the difference $\chi_k(K(m,n)\Box K(m,n)) - \chi_k(K(m,n))$ can be as large as desired, even when $k=2$. We also show that there are families of graphs for which equality (1) holds for k-tuple colorings of graphs for any $k \geq 1$. As far as we know, our results are the first ones concerning the k-tuple chromatic number of cartesian product of graphs.

2 Cartesian products of Kneser graphs

We start this section with some upper and lower bounds for the k-tuple chromatic number of Kneser graphs.

Lemma 1. *Let G be a graph and let $k > 0$. Then, $\chi_k(G\Box G) \leq k\chi(G)$.*

Proof. Clearly, $\chi_k(G\Box G) \leq k\chi(G\Box G)$. However, by equality (1) we know that $\chi(G\Box G) = \chi(G)$, and thus the lemma holds. $\Box$

Corollary 1. $\chi_k(K(m,n)\Box K(m,n)) \leq k\chi(K(m,n)) = k(m-2n+2)$.

We can obtain a trivial lower bound for the k-tuple chromatic number of the graph $K(m,n)\Box K(m,n)$ in terms of the clique number of $K(m,n)$. In fact, notice that $\omega(K(m,n)\Box K(m,n)) = \omega(K(m,n)) = \lfloor \frac{m}{n} \rfloor$. Thus, we have that $\chi_k(K(m,n)\Box K(m,n)) \geq k\omega(K(m,n)) = k\lfloor \frac{m}{n} \rfloor$.

Larose et al. [8] showed that no connected Kneser graph $K(m,n)$ is hom-idempotent, that is, for any $m > 2n$, there is no homomorphism from $K(m,n)\Box K(m,n)$ to $K(m,n)$.

Lemma 2 ([8])**.** *Let $m > 2n$. Then, $K(m,n)\Box K(m,n) \not\to K(m,n)$.*

Concerning the k-tuple chromatic number of some Kneser graphs, Stahl [11] showed the following results.

Lemma 3 ([11]). *If $1 \leq k \leq n$, then $\chi_k(K(m,n)) = m - 2(n-k)$.*

Lemma 4 ([11]). *$\chi_k(K(2n+1,n)) = 2k + 1 + \lfloor \frac{k-1}{n} \rfloor$, for $k > 0$.*

Lemma 5 ([11]). *$\chi_{rn}(K(m,n)) = rm$, for $r > 0$ and $m \geq 2n$.*

By using Lemma 5 we have the following result.

Lemma 6. *Let $m > 2n$. Then, $\chi_n(K(m,n)\Box K(m,n)) > \chi_n(K(m,n))$.*

Proof. By Lemma 5 when $r = 1$, we have that $\chi_n(K(m,n)) = m$. If $\chi_n(K(m,n)\Box K(m,n)) = m$, then there exists an homomorphism from the graph $K(m,n)\Box K(m,n)$ to $K(m,n)$ which contradicts Lemma 2. $\Box$

By Lemma 3, Lemma 6 and by using Corollary 1, we have that,

Corollary 2. *Let $n \geq 2$. Then, $2n + 2 \leq \chi_n(K(2n+1,n)\Box K(2n+1,n)) \leq 3n$. In particular, when $n = 2$, we have that $\chi_2(K(5,2)\Box K(5,2)) = 6$.*

In the case $k = 2$ we have by Lemma 6, Lemma 3 and by Corollary 1, the following result.

Corollary 3. *Let $q > 0$. Then, $q + 4 \leq \chi_2(K(2n+q,n)\Box K(2n+q,n)) \leq 2q + 4$.*

By Corollary 3, notice that in the case when $k = n = 2$ and $q \geq 1$, we must have that $\chi_2(K(q+4,2)\Box K(q+4,2)) > q+4$, otherwise there is a contradiction with Lemma 2. This provides a gap of one unity between the 2-tuple chromatic number of the graph $K(q+4,2)\Box K(q+4,2)$ and the graph $K(q+4,2)$. In the following, we will prove that, for some Kneser graphs, such a gap can be as large as desired. In order to do this, we need the following technical tools.

A *stable set* $S \subseteq V$ is a subset of pairwise non adjacent vertices of G. The *stability number* of G, denoted by $\alpha(G)$, is the largest cardinality of a stable set in G. Let $m \geq 2n$. An element $i \in [m]$ is called a *center* of a stable set S of the Kneser graph $K(m,n)$ if it lies in each n-set in S.

Lemma 7 (Erdős-Ko-Rado [5]). *If $m > 2n$, then $\alpha(K(m,n)) = \binom{m-1}{n-1}$. A stable set of $K(m,n)$ with size $\binom{m-1}{n-1}$ has a center i, for some $i \in [m]$.*

Lemma 8 (Hilton-Milner [7]). *If $m \geq 2n$, then the maximum size of a stable set in $K(m,n)$ with no center is equal to $1 + \binom{m-1}{n-1} - \binom{m-n-1}{n-1}$.*

A graph $G = (V,E)$ is *vertex transitive* if its automorphism group acts transitively on V, that is, for any pair of distinct vertices of G there is an automorphism mapping one to the other one. It is well known that Kneser graphs are vertex transitive graphs.

Lemma 9 (No-Homomorphism Lemma, Albertson-Collins [1]). *Let G, H be graphs such that H is vertex transitive and $G \to H$. Then,*

$$\alpha(G)/|V(G)| \geq \alpha(H)/|V(H)|.$$

Lemma 10. *Let $m > 2n$. Then, $\chi_k(K(m,n)\Box K(m,n)) \geq k\frac{\binom{m}{n}^2}{\alpha(K(m,n)\Box K(m,n))}$.*

Proof. Let $t = \chi_k(K(m,n)\Box K(m,n))$. Then, $K(m,n)\Box K(m,n) \to K(t,k)$ and from the No-Homomorphism Lemma, $\frac{\alpha(K(m,n)\Box K(m,n))}{|V(K(m,n)\Box K(m,n))|} \geq \frac{\alpha(K(t,k))}{|V(K(t,k))|}$. The result follows from the fact that $\frac{\alpha(K(t,k))}{|V(K(t,k))|} = \frac{k}{t}$. □

An *edge-coloring* of a graph $G = (V, E)$ is an assignment of colors to the edges of G such that any two incident edges are assigned different colors. The smallest number t such that G admits an edge-coloring with t colors is called the *chromatic index* of G and is denoted by $\chi'(G)$. It is well known that the chromatic index of a complete graph K_n on n vertices is equal to $n-1$ if n is even and n if n is odd (see [2]). Besides, in the case n even each color class i (i.e. the subset of pairwise non incident edges colored with color i) has size $\frac{n}{2}$ and if n is odd each color class has size $\frac{n-1}{2}$. Therefore, using this fact, we obtain the following result.

Lemma 11. *Let $q \geq 5$. If q is even then the set of vertices of the Kneser graph $K(q,2)$ can be partitioned into $q-1$ disjoint cliques, each one with size $\frac{q}{2}$ and if q is odd then the set of vertices of the Kneser graph $K(q,2)$ can be partitioned into q disjoint cliques, each one with size $\frac{q-1}{2}$.*

Proof. Notice that there is a natural bijection between the vertex set of $K(q,2)$ and the edge set of the complete graph K_q with vertex set $[q]$: each vertex $\{i,j\}$ in $K(q,2)$ is mapped to the edge $\{i,j\}$ in K_q. Now, if q is even there is a $(q-1)$-edge coloring of K_q where each color class is a set of pairwise non incident edges with size $\frac{q}{2}$ and if q is odd there is a q-edge coloring of K_q where each color class is a set of pairwise non incident edges with size $\frac{q-1}{2}$. Notice that two edges $e, e' \in K_q$ are non incident edges if and only if $e \cap e' = \emptyset$. Therefore, a color class of the edge-coloring of K_q represents a clique of $K(q,2)$. □

Now, we are able to obtain an upper bound for the stability number of the graph $K(q,2)\Box K(q,2)$ as follows.

Lemma 12. *Let $q \geq 5$. Then,*

- $\alpha(K(q,2)\Box K(q,2)) \leq \frac{q(q-1)}{8}(3q-2)$ *if q is even and,*
- $\alpha(K(q,2)\Box K(q,2)) \leq \frac{q(q-1)}{8}(3q-1)$ *if q is odd.*

Proof. Let q even. First, recall that a stable set X in $K(q,2)$ has size at most $q-1$ if X has center (see Lemma 7) and $|X| \leq 1 + (q-1) - (q-2-1) = 3$ if X has no center (see Lemma 8). Besides, observe that the vertex set of $K(q,2)$ can be partitioned in $q-1$ sets $S_1, \ldots, S_{q-1}$ such that each S_i induces a complete subgraph graph $K_{\frac{q}{2}}$ in $K(q,2)$, for $i = 1, \ldots, q-1$ (see Lemma 11). Consider the subgraph H_i of $K(q,2)\Box K(q,2)$ induced by $S_i \times V(K(q,2))$ for $i = 1, \ldots, q-1$. Let I be a stable set in $K(q,2)\Box K(q,2)$ and $I_i = I \cap H_i$ for $i = 1, \ldots, q-1$. Then, for each $v \in S_i$, $I_i^v = I_i \cap (\{v\} \times V(K(q,2)))$ is a stable set in $K(q,2)\Box K(q,2)$ for each $i = 1, \ldots, q-1$. Finally, for each $m \in S_i$, with $1 \leq i \leq q-1$, let $I_{i,2}^m$ be the stable set in $K(q,2)$ such that $I_i^m = \{m\} \times I_{i,2}^m$.

Now, for a fixed $i \in \{1, \ldots, q-1\}$, assume w.l.o.g. that r ($r \leq \frac{q}{2}$) stable sets $I_{i,2}^1, \ldots, I_{i,2}^r$ of $K(q,2)$ have distinct center $j_1, \ldots, j_r$, respectively (the case when two of these stable sets have the same center can be easily reduced to this case). Let W be the set of subsets with size two of $\{j_1, \ldots, j_r\}$. Therefore, for all $m \in \{1, \ldots, r\}$, $I_i^m - (\{m\} \times W)$ has size at most $q-1-(r-1) = q-r$ since each center j_m belongs to $r-1$ elements in W. Besides, each element of W belongs to exactly one set $I_{i,2}^m$ for $m \in \{1, \ldots, r\}$, since S_i induces a complete subgraph in $K(q,2)$ and $\{1, \ldots, r\} \subseteq S_i$.

Then, $|I_i^1 \cup \ldots \cup I_i^r| \leq (\sum_{m=1}^r |I_i^m - \{m\} \times W|) + |W| \leq r(q-r) + \frac{r(r-1)}{2}$. Next, each remaining stable set (if exist) $I_{i,2}^{r+1}, \ldots, I_{i,2}^{\frac{q}{2}}$ has no center, then $|I_i^d| \leq 3$ for all $d \in \{r+1, \ldots, \frac{q}{2}\}$. Thus, $|I_i| \leq r(q-r) + \frac{r(r-1)}{2} + 3(\frac{q}{2} - r) = -\frac{r^2}{2} + r(q - \frac{7}{2}) + \frac{3}{2}q$. Since the last expression is non decreasing for $r \in \{1, \ldots, \frac{q}{2}\}$, we have that

$$|I_i| \leq -\frac{q^2}{8} + \frac{q}{2}(q - \frac{7}{2}) + 3\frac{q}{2} = \frac{q}{2}(\frac{3}{4}q - \frac{1}{2})$$

Therefore, $|I_i| \leq \frac{q}{2}(\frac{3}{4}q - \frac{1}{2})$ for every $i = 1, \ldots, q-1$. Since $|I| = \sum_{i=1}^{q-1} |I_i|$, it follows that $|I| \leq \frac{q(q-1)}{2}(\frac{3}{4}q - \frac{1}{2})$ and thus,

$$\alpha(K(q,2) \square K(q,2)) \leq \frac{q(q-1)}{8}(3q-2)$$

We analyze now the case for q odd, with a similar reasoning. First, recall that a stable set X in $K(q,2)$ has size at most $q-1$ if X has center (see Lemma 7) and $|X| \leq 1 + (q-1) - (q-2-1) = 3$ if X has no center (see Lemma 8). Besides, observe that the vertex set of $K(q,2)$ can be partitioned in q sets $S_1, \ldots, S_q$ such that each S_i induces a complete subgraph $K_{\frac{q-1}{2}}$ in $K(q,2)$, for $i = 1, \ldots, q$ (see Lemma 11). Consider the subgraph H_i of $K(q,2) \square K(q,2)$ induced by $S_i \times V(K(q,2))$ for $i = 1, \ldots, q$. Let I be a stable set in $K(q,2) \square K(q,2)$ and $I_i = I \cap H_i$ for $i = 1, \ldots, q$. Then, for each $v \in S_i$, $I_i^v = I_i \cap (\{v\} \times V(K(q,2)))$ is a stable set in $K(q,2) \square K(q,2)$ for each $i = 1, \ldots, q$. Finally, for each $m \in S_i$, with $1 \leq i \leq q$, let $I_{i,2}^m$ be the stable set in $K(q,2)$ such that $I_i^m = \{m\} \times I_{i,2}^m$.

Now, for a fixed $i \in \{1, \ldots, q\}$, assume w.l.o.g. that r ($r \leq \frac{q-1}{2}$) stable sets $I_{i,2}^1, \ldots, I_{i,2}^r$ of $K(q,2)$ have distinct center $j_1, \ldots, j_r$, respectively (the case when two of these stable sets have the same center can be easily reduced to this case). Let W be the set of subsets with size two of $\{j_1, \ldots, j_r\}$. Therefore, for all $m \in \{1, \ldots, r\}$, $I_i^m - (\{m\} \times W)$ has size at most $q-1-(r-1) = q-r$ since each center j_m belongs to $r-1$ elements in W. Besides, each element of W belongs to exactly one set I_i^m for $m \in \{1, \ldots, r\}$, since S_i induces a complete subgraph in $K(q,2)$ and $\{1, \ldots, r\} \subseteq S_i$. Then, $|I_i^1 \cup \ldots \cup I_i^r| \leq (\sum_{m=1}^r |I_i^m - \{m\} \times W|) + |W| \leq r(q-r) + \frac{r(r-1)}{2}$.

Next, each remaining stable set (if exist) $I_{i,2}^{r+1}, \ldots, I_{i,2}^{\frac{q-1}{2}}$ has no center, then $|I_i^d| \leq 3$ for all $d \in \{r+1, \ldots, \frac{q-1}{2}\}$. Thus, $|I_i| \leq r(q-r) + \frac{r(r-1)}{2} + 3(\frac{q-1}{2} - r) = -\frac{r^2}{2} + r(q - \frac{7}{2}) + \frac{3}{2}(q-1)$. Since the last expression is non decreasing for $r \in \{0, \ldots, \frac{q-1}{2}\}$, we have that

$$|I_i| \leq -\frac{(q-1)^2}{8} + \frac{q-1}{2}(q - \frac{7}{2}) + \frac{3}{2}(q-1) = \frac{q-1}{2}(\frac{3}{4}q - \frac{1}{4})$$

Therefore, $|I_i| \leq \frac{q-1}{2}(\frac{3}{4}q - \frac{1}{4})$ for every $i = 1, \ldots, q$. Since $|I| = \sum_{i=1}^{q} |I_i|$, it follows that $|I| \leq \frac{q(q-1)}{2}(\frac{3}{4}q - \frac{1}{4})$ and thus,

$$\alpha(K(q,2) \square K(q,2)) \leq \frac{q(q-1)}{8}(3q-1)$$

□

From Lemmas 10 and 12 we have the following result.

Theorem 1. *Let $q \geq 5$. Then,*

- $\chi_k(K(q,2) \square K(q,2)) \geq 2k\frac{q(q-1)}{3q-2}$ *if q is even and,*

- $\chi_k(K(q,2)\Box K(q,2)) \geq 2k\frac{q(q-1)}{3q-1}$ *if q is odd.*

In the particular case when $q = 2s+4$, with $s > 0$, and $k = 2$, we have, by Lemma 5 and Theorem 1, the following result that shows that the difference $\chi_2(K(2s+4,2)\Box K(2s+4,2)) - \chi_2(K(2s+4,2))$ can be as large as desired.

Corollary 4. *For any integer $s > 0$ and for $k = 2$, we have that,*

$$\chi_2(K(2s+4,2)\Box K(2s+4,2)) \geq 2s + \left\lceil \frac{2}{3}s \right\rceil + 5 = \chi_2(K(2s+4,2)) + \left\lceil \frac{2}{3}s \right\rceil + 1.$$

From Lemmas 4 and 5, Corollary 1, and Theorem 1, we obtain the results that we summarize in Table 1.

G	k	$\chi_k(G)$	$\chi_k(G\Box G) =$	$\chi_k(G\Box G) \geq$	$\chi_k(G\Box G) \leq$
$K(5,2)$	2	5	6	-	-
-	3	8	9	-	-
-	4	10	12	-	-
-	5	13	15	-	-
-	6	15	18	-	-
-	7	18	?	20	21
$K(6,2)$	2	6	8	-	-
-	3	?	12	-	-
-	4	12	?	15	16
-	5	?	?	19	20
$K(7,2)$	2	7	?	9	10
-	3	?	?	13	15
$K(8,2)$	2	8	?	11	12
-	3	?	?	16	18

Table 1: Summary of results

Finally, by applying some known homomorphisms between Kneser graphs, we obtain the following result.

Theorem 2. *Let $k > n$ and let $t = \chi_k(K(m,n)\Box K(m,n))$, where $m > 2n$. Then, either $t > m + 2(k-n)$ or $t < m + (k-n)$.*

Proof. Suppose that $m + (k-n) \leq t \leq m + 2(k-n)$. Therefore, there exists an homomorphism $K(m,n)\Box K(m,n) \to K(t,k)$. Now, Stahl [11] showed that there is an homomorphism $K(m,n) \to K(m-2,n-1)$ whenever $n > 1$ and $m \geq 2n$. Moreover, it is easy to see that there is an homomorphism $K(m,n) \to K(m-1,n-1)$. By applying the former homomorphism $t-(m+(k-n))$ times to the graph $K(t,k)$ we obtain an homomorphism $K(t,k) \to K(2(m+k-n)-t, 2k+m-n-t)$. Finally, by applying $2k+m-t-2n$ times the latter homomorphism to the graph $K(2(m+k-n)-t, 2k+m-n-t)$ we obtain an homomorphism $K(2(m+k-n)-t, 2k+m-n-t) \to K(m,n)$. Therefore, by homomorphism composition, $K(m,n)\Box K(m,n) \to K(m,n)$ which contradicts Lemma 2. $\square$

3 Cases where $\chi_k(G\Box H) = \max\{\chi_k(G), \chi_k(H)\}$

Theorem 3. *Let G and H be graphs such that $\chi(G) \leq \chi(H) = \omega(H)$. Then, $\chi_k(G\Box H) = \max\{\chi_k(G), \chi_k(H)\}$.*

Proof. Let $t = \omega(H)$ and let $\{h_1, \ldots, h_t\}$ be the vertex set of a maximum clique K_t in H with size t. Clearly, $\chi_k(G) \leq \chi_k(H) = \chi_k(K_t)$. Let ρ be a k-tuple coloring of H with $\chi_k(H)$ colors. By equality (1), there exists a t-coloring f of $G\Box H$. Therefore, the assignment of the k-set $\rho(h_{f((a,b))})$ to each vertex (a, b) in $G\Box H$ defines a k-tuple coloring of $G\Box H$ with $\chi_k(K_t)$ colors. $\Box$

Notice that if G and H are both bipartite, then $\chi_k(G\Box H) = \chi_k(G) = \chi_k(H)$. In the case when G is not a bipartite graph, we have the following results.

An *automorphism* σ of a graph G is called a *shift* of G if $\{u, \sigma(u)\} \in E(G)$ for each $u \in V(G)$ [8]. In other words, a shift of G maps every vertex to one of its neighbors.

Theorem 4. *Let G be a non bipartite graph having a shift $\sigma \in AUT(G)$, and let H be a bipartite graph. Then, $\chi_k(G\Box H) = \max\{\chi_k(G), \chi_k(H)\}$.*

Proof. Let $A \cup B$ be a bipartition of the vertex set of H. Let f be a k-tuple coloring of G with $\chi_k(G)$ colors. Clearly, $\chi_k(G) \geq \chi_k(H)$. We define a k-tuple coloring ρ of $G\Box H$ with $\chi_k(G)$ colors as follows: for any vertex (u, v) of $G\Box H$ with $u \in G$ and $v \in H$, define $\rho((u, v)) = f(u)$ if $v \in A$, and $\rho((u, v)) = f(\sigma(u))$ if $v \in B$. $\Box$

We may also deduce the following direct result.

Theorem 5. *Let G be an hom-idempotent graph an let H be a subgraph of G. Thus, $\chi_k(G\Box H) = \max\{\chi_k(G), \chi_k(H)\} = \chi_k(G)$.*

Let A be a group and S a subset of A that is closed under inverses and does not contain the identity. The *Cayley graph* $Cay(A, S)$ is the graph whose vertex set is A, two vertices u, v being joined by an edge if $u^{-1}v \in S$. If $a^{-1}Sa = S$ for all $a \in A$, then $Cay(A, S)$ is called a *normal Cayley graph.*

Lemma 13 ([6])**.** *Any normal Cayley graph is hom-idempotent.*

Note that all Cayley graphs on Abelian groups are normal, and thus hom-idempotent. In particular, the *circulant* graphs are Cayley graphs on cyclic groups (i.e., cycles, powers of cycles, complements of powers of cycles, complete graphs, etc). By Theorem 5 and Lemma 13 we have the following result.

Theorem 6. *Let $Cay(A, S)$ be a normal Cayley graph and let $Cay(A', S')$ be a subgraph of $Cay(A, S)$, with $A' \subseteq A$ and $S' \subseteq S$. Then,*
$\chi_k(Cay(A, S)\Box Cay(A', S')) = \max\{\chi_k(Cay(A, S)), \chi_k(Cay(A', S'))\}$.

Definition 1. *Let G be a graph with a shift σ. We define the* order *of σ as the minimum integer i such that σ^i is equal to the identity permutation.*

Theorem 7. *Let G be a graph with a shift σ of minimum odd order $2s+1$ and let C_{2t+1} be a cycle graph, where $t \geq s$. Then,*
$\chi_k(G\Box C_{2t+1}) = \max\{\chi_k(G), \chi_k(C_{2t+1})\}$.

Proof. Let $\{0, \ldots, 2t\}$ be the vertex set of C_{2t+1}, where for $0 \leq i \leq 2t$, $\{i, i+1 \mod (2t+1)\} \in E(C_{2t+1})$. Let G_i be the i^{th} copy of G in $G \Box C_{2t+1}$, that is, for each $0 \leq i \leq 2t$, $G_i = \{(g, i) : g \in G\}$. Let f be a k-tuple coloring of G with $\chi_k(G)$ colors. We define a k-tuple coloring of $G \Box C_{2t+1}$ with $\chi_k(G)$ colors as follows: let σ^0 denotes the identity permutation of the vertices in G. Now, for $0 \leq i \leq 2s$, assign to each vertex $(u, i) \in G_i$ the k-tuple $f(\sigma^i(u))$. For $2s+1 \leq j \leq 2t$, assign to each vertex $(u, j) \in G_j$ the k-tuple $f(u)$ if j is odd, otherwise, assign to (u, j) the k-tuple $f(\sigma^1(u))$. It is not difficult to see that this is in fact a proper k-tuple coloring of $G \Box C_{2t+1}$. $\square$

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