The Permutation-Path Coloring Problem on Trees

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Abstract

In this paper we first show that the permutation-path coloring problem is NP-hard even for very restrictive instances like *involutions*, which are permutations that contain only cycles of length at most two, on both binary trees and on trees having only two vertices with degree greater than two, and for *circular permutations*, which are permutations that contain exactly one cycle, on trees with maximum degree greater or equal to 4. We calculate a lower bound on the average complexity of the permutation-path coloring problem on arbitrary networks. Then we give combinatorial and asymptotic results for the permutation-path coloring problem on linear networks in order to show that the average number of colors needed to color any permutation on a linear network on n vertices is $n/4 + o(n)$. We extend these results and obtain an upper bound on the average complexity of the permutation-path coloring problem on arbitrary trees, obtaining exact results in the case of generalized star trees. Finally we explain how to extend these results for the involutions-path coloring problem on arbitrary trees.

Keywords: Average-Case Complexity, Routing Permutation, Path Coloring, Tree Networks, NP-completeness.

1 Introduction

Efficient communication is a prerequisite to exploit the performance of large parallel systems. The routing problem on communication networks consists in the efficient allocation of resources to connection requests. In this network, establishing a connection between two nodes requires *selecting* a path connecting the two nodes and *allocating* sufficient resources on all links along the paths associated to the collection of requests. In the case of *all-optical* networks, data is transmitted on lightwaves through optical fiber, and several signals can be transmitted through a fiber link simultaneously provided that different wavelengths are used in order to prevent interference (wavelength-division multiplexing) [5]. As the number of wavelengths is a limited resource, it is desirable to establish a given set of connection requests with a minimum number of wavelengths.

The routing problem for all-optical networks can be viewed as a path coloring problem: it consists in finding a desirable collection of paths on the network associated with the collection of connection requests in order to minimize the number of colors needed to color these paths in such a way that any two different paths sharing a link of the network are assigned different colors. For simple networks, such as trees, the routing problem is simpler, as there is a unique path for each communication request.

This paper deals with the problem of routing a set of communication requests representing a permutation of the nodes of an all-optical tree network using the *wavelength division multiplexing* (or WDM) technology. Clearly, such a routing problem can be modeled as a permutation-path coloring problem on trees. An instance of the permutation-path coloring problem on trees is given by a directed symmetric tree graph T on n nodes and a permutation σ of the vertex set of T. Moreover, we associate with each pair $(i, \sigma(i)), i \neq \sigma(i), 1 \leq i \leq n$, the unique directed path on T from vertex i to vertex $\sigma(i)$. Thus, the permutation-path coloring problem for this instance consists in assigning the minimum number of colors to such a permutation-set of paths in such a way that any two paths sharing an arc of the tree are assigned different colors. In fact, the colors in the latter problem represent the wavelengths in the former one.

Related work. Using a result of Leighton and Rao [21], Aumann and Rabani [1] show that $O(\frac{\log^2 n}{\beta^2})$ colors suffice for routing any permutation on any bounded degree network on n nodes, where β is the *arc expansion* of the network. The result of Aumman and Rabani almost matches the existential lower bound of $\Omega(\frac{1}{\beta^2})$ obtained by Raghavan and Upfal [25]. In the case of specific network topologies, Gu and Tamaki [17] prove that 2 colors are sufficient to route any permutation on any symmetric directed hypercube. Independently, Paterson et al. [24] and Wilfong and Winkler [28] show that the routing permutation problem on ring networks is NP-hard. Moreover, in [28] a 2-approximation algorithm is given for this problem on ring networks. Independently, Kumar et al. [19] and Erlebach and Jansen [8] show that computing a minimal coloring of any collection of paths on binary trees is NP-hard. Caragiannis et al. [4] consider the symmetric-path coloring problem on trees (i.e., for each path from vertex u to vertex v, there also exists its symmetric, a path from vertex v to vertex u) showing that this special instance is also NP-hard for unbounded degree trees, and leaving as an open problem the complexity of such a symmetric instances on binary trees. To our knowledge, the routing permutation problem on arbitrary tree networks by arc-disjoint paths has not been studied in the literature.

Our results. In Section 3 we show that the symmetric-path coloring problem on binary trees is NP-hard, answering an open question in [4]. Moreover, we extend such a result in order to show that the permutation-path coloring problem remains NP-hard even for very restrictive instances like *involutions*, which are permutations that contain only cycles of length at most two, both on binary trees and on trees having only two vertices with degree greater than two, and for *circular permutations*, which are permutations that contain exactly one cycle, on trees with maximum degree greater or equal to 4. In Section 4 we compute a lower bound for the average number of colors needed to color any permutation-path set on arbitrary networks. In Section 5 we focus on linear networks. In this particular case, since the problem reduces to coloring an interval graph [16], the routing of any permutation is easily done in polynomial time [18]. We show that the average number of colors needed to color any permutation-path set on a linear network on n vertices is $n/4 + o(n)$. In Section 6, we extend the results obtained in Section 5, by giving an upper bound on the average number of colors needed to color any permutation-path set on arbitrary trees, obtaining exact results in the case of generalized star tree networks. As far as we know, this is the first result on the average-case complexity for routing permutations on all-optical networks. Finally we show how to extend these results to the involution problem partly studied in [20].

2 Definitions and preliminary results

We model the tree network as a rooted labeled symmetric directed tree $T = (V, A)$ on n vertices, where processors and switches are vertices and links are modeled by two arcs in opposite directions. Let $\mathcal{P}$ be a collection of directed paths on T. We assume that the vertices of T are arbitrarily labeled by different integers in $\{1, 2, \ldots, n\}$ and that vertex labeled with the integer n is the root vertex of T. We denote by $i \rightsquigarrow j$ the unique directed path from vertex i to vertex j in T. The arc from vertex i to its father (resp. from the father of i to i), $1 \leq i \leq n-1$, is labeled by i^+ (resp. i^-). We call $T(i)$ the subtree of T rooted at vertex i, $1 \leq i \leq n$. See Figure 1(a) for the linear network on $n = 6$ vertices rooted at vertex $i = 6$. Note that we will just draw an edge i rather than the arcs i^+ and i^- in the sequel.

For any i, $1 \leq i \leq n-1$, the *load* of an arc i^+ (resp. i^-) of T, denoted by $L_T(\mathcal{P}, i^+)$ (resp. $L_T(\mathcal{P}, i^-)$), is the number of paths in $\mathcal{P}$ using such an arc, and the *maximum load* among all arcs of T is denoted by $L_T(\mathcal{P})$, i.e., $L_T(\mathcal{P}) = \max_i(\max(L_T(\mathcal{P}, i^+), L_T(\mathcal{P}, i^-)))$. We call *the coloring number* and we denote by $R_T(\mathcal{P})$, the minimum number of colors needed to color the paths in $\mathcal{P}$ such that any two paths sharing an arc in T are assigned different colors. Trivially, we have that $R_T(\mathcal{P}) \geq L_T(\mathcal{P})$. Let $\mathcal{S}_n$ denote the symmetric group of all permutations on $[n] = \{1, 2, \ldots, n\}$. Let σ be a permutation in $\mathcal{S}_n$. Then σ is called *an involution* (resp. *a circular permutation*) if it contains only cycles of length at most two

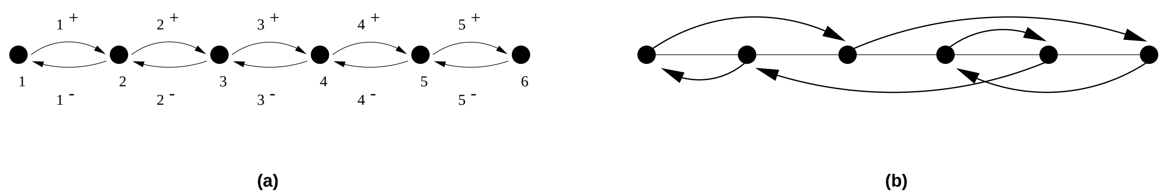


Figure 1: (a) Labeling of the vertices and the arcs for the linear network on $n = 6$ vertices rooted at vertex $i = 6$. (b) Representation of the permutation $\sigma = (3, 1, 6, 5, 2, 4)$ on the linear network given in (a).

(resp. contains exactly one cycle of length n). Let I_{2n} be the set of involutions with no fixed point on $[2n]$. We say that $\mathcal{P}$ is a *permutation-path* set on T if $\mathcal{P}$ represents a permutation $\sigma \in \mathcal{S}_n$ of the vertex-set of T, where $\sigma(i) = j$, $i \neq j$, if and only if $i \rightsquigarrow j \in \mathcal{P}$. In the sequel we talk indifferently of a permutation-path set $\mathcal{P}$ or of the permutation $\sigma \in \mathcal{S}_n$ that $\mathcal{P}$ represents. Thus, given a permutation $\sigma \in \mathcal{S}_n$ and a tree T on n vertices, the load of the arc i^+ (resp. i^-), $1 \leq i \leq n-1$, can be expressed by $L_T(\sigma, i^+) = |\{j \in T(i) : \sigma(j) \notin T(i)\}|$ (resp. $L_T(\sigma, i^-) = |\{j \notin T(i) : \sigma(j) \in T(i)\}|$).

Lemma 1 *Let T be a tree on n vertices. For all $\sigma \in \mathcal{S}_n$ and for all $i \in \{1, 2, \ldots, n-1\}$, $L_T(\sigma, i^+) = L_T(\sigma, i^-)$. Therefore, $L_T(\sigma) = \max_i L_T(\sigma, i^+)$.*

Proof. We can prove this by induction on the height of the vertices. If vertex i is a leaf, we have two cases:

- $\sigma(i) = i$, then $L_T(\sigma, i^+) = L_T(\sigma, i^-) = 0$,
- $\sigma(i) \neq i$, then $L_T(\sigma, i^+) = L_T(\sigma, i^-) = 1$.

Otherwise, let vertex i be an internal vertex. Let $\{i_1, i_2, \ldots, i_j\}$ be the sons of the vertex i, and $N(i_j)$ be the number of vertices $k \in T(i_j)$ such that $\sigma(k) \notin T(i_j)$ and $\sigma(k) \in T(i)$. Then it is easy to see that $L_T(\sigma, i^+)$ and $L_T(\sigma, i^-)$ satisfy the same recurrence relation for any internal vertex i :

$$L_T(\sigma, i^{\pm}) = \begin{cases} \sum_{k=1}^{j} L_T(\sigma, i_k^{\pm}) - N(i_k) & \text{if } \sigma(i) = i \text{ or } (\sigma(i) \in T(i) \text{ and } \sigma^{-1}(i) \notin T(i)) \\ & \text{or } (\sigma(i) \notin T(i) \text{ and } \sigma^{-1}(i) \in T(i)) \\ 1 + \sum_{k=1}^{j} L_T(\sigma, i_k^{\pm}) - N(i_k) & \text{if } \sigma(i) \notin T(i) \text{ and } \sigma^{-1}(i) \notin T(i) \\ -1 + \sum_{k=1}^{j} L_T(\sigma, i_k^{\pm}) - N(i_k) & \text{if } \sigma(i) \in T(i) \text{ and } \sigma^{-1}(i) \in T(i) \end{cases} \tag{1}$$

$\square$

This lemma tells us that in order to study the load of a permutation on a tree on n vertices, it suffices to consider the load of the labeled arcs i^+. For example the permutation $\sigma = (3,1,6,5,2,4)$ on the linear network in Figure 1(b) has load 2. The maximum is reached in the arcs $4^\pm$.

Definition 1 *Let T be a tree on n vertices. The average load of all permutations $\sigma \in \mathcal{S}_n$ on T, denoted by $\bar{L}_T$, is defined as $\bar{L}_T = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} L_T(\sigma)$.*

Proposition 1 *[9] There is a polynomial-time algorithm to color any collection $\mathcal{P}$ of paths on any tree T such that $L_T(\mathcal{P}) \le R_T(\mathcal{P}) \le \lceil \frac{5}{3} L_T(\mathcal{P}) \rceil$.*

Given a tree T on n vertices, we denote by $\bar{R}_T$ the average number of colors needed to color all permutations in $\mathcal{S}_n$ on T. Thus, by Proposition 1, we have the following lemma.

Lemma 2 *Let T be a tree on n vertices. Then $\bar{L}_T \le \bar{R}_T \le \frac{5}{3}\bar{L}_T + 1$.*

Proposition 2 *[4] There is a polynomial-time algorithm to color any collection $\mathcal{P}$ of symmetric paths on any tree T such that $L_T(\mathcal{P}) \le R_T(\mathcal{P}) \le \lfloor \frac{3}{2} L_T(\mathcal{P}) \rfloor$.*

Given a tree T on $2n$ vertices, we denote by $\tilde{R}_T$ the average number of colors needed to color all involutions in I_{2n} on T. Thus, by Proposition 2, we have the following lemma.

Lemma 3 *Let T be a tree on $2n$ vertices and let $\tilde{L}_T$ be the average load of all involutions in I_{2n} on T. Then $\tilde{L}_T \le \tilde{R}_T \le \frac{3}{2}\tilde{L}_T$.*

Definition 2 *Let T be a tree and let $\mathcal{P}$ be a collection of paths on T. The conflict graph, denoted by $G_T(\mathcal{P}) = (V, E)$, is the undirected graph associated with T and $\mathcal{P}$, where each vertex $v_p \in V$ represents a path $p \in \mathcal{P}$, and two vertices v_p and v_q are joined by an edge in E if and only if their associated paths p and q share a same arc in T.*

It is straightforward to see that the coloring number $R_T(\mathcal{P})$ is equal to the chromatic number of the conflict graph $G_T(\mathcal{P})$.

Definition 3 *Let T be a tree and let $\mathcal{P}$ be a collection of paths on T. The digraph associated with $\mathcal{P}$, denoted $\vec{G}_T(\mathcal{P})$, is the digraph with vertex set V', where $v \in V'$ if and only if v is a vertex of T and there is at least one path in $\mathcal{P}$ having v as ending-vertex, and with arc set $A' = \{(v,w) : v, w \in V' \text{ and } v \rightsquigarrow w \in \mathcal{P}\}$.*

A digraph $G = (V, A)$ is said to be *pseudo-symmetric*, if for any vertex $v \in V$, $d^+(v) = d^-(v)$, where $d^+(v)$ (resp. $d^-(v)$) denotes the out-degree (resp. in-degree) of vertex v.

Theorem 1 *[11] If G is a connected pseudo-symmetric digraph, then G is Eulerian and an Eulerian circuit of G can be found in linear time.*

Let P_n denote the directed symmetric path graph on n vertices. Let $\mathrm{ST}(n)$ denote the directed symmetric star graph on n vertices (i.e., the tree having only one internal vertex connected to $n-1$ leaves). We call *generalized star* graph that we denote by $\mathrm{GST}(\lambda)$, a directed symmetric graph on n vertices having k branches connected to each other by one vertex, where $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of the integer $n-1$ into k parts ($k > 2$) and where λ_i denotes the length of the i^{th} branch (i.e., a branch of length λ_i is a path graph on $\lambda_i + 1$ vertices).

3 Complexity of computing the coloring number

We begin this section by showing the NP-hardness of the symmetric-path coloring problem on binary trees, answering an open question in [4]. Moreover, we extend this result by showing that the permutation-path coloring problem remains NP-hard even for very restrictive instances like *involutions* on both binary trees and on trees having only two vertices with degree greater than two, and for *circular permutations* on trees with maximum degree greater or equal to 4. Finally, we discuss some polynomial cases of this problem.

3.1 NP-hardness results

This section shows that the path coloring problem on trees is difficult even for very restrictive cases. For this, we use a reduction similar to the one used in [8, 19] for proving the NP-hardness of the general path coloring problem on binary trees. We remark that the reduction used in [8, 19] cannot be directly extended to obtain NP-hardness results on the restrictive instances of the problem that we consider in the following theorem.

Theorem 2 *Let T be a directed symmetric tree and let $\mathcal{P}$ be a collection of directed paths on T. Then, computing $R_T(\mathcal{P})$ is NP-hard in the following cases:*

(a) T is a binary tree and $\mathcal{P}$ is a collection of symmetric paths on T.
(b) T is a binary tree and $\mathcal{P}$ represents an involution of the vertices of T.
(c) T is a tree with maximum degree greater or equal to 4, and $\mathcal{P}$ represents a circular permutation of the vertices of T.
(d) T is a tree having only two vertices with degree greater than two and $\mathcal{P}$ represents an involution of the vertices of T.

Proof. We use a reduction from the ARC-COLORING problem [26]. The ARC-COLORING problem is defined as follows: we are given a positive integer k, an undirected cycle C_n with vertex set numbered clockwise as $1, 2, \ldots, n$, and any collection of paths F on C_n, where each path in F from vertex v to vertex w, denoted by $< v, w >$, is regarded as the path beginning at vertex v and ending at vertex w again in the clockwise direction. The question is, can F be colored with k colors such that no two paths sharing an edge of C_n are assigned the same color ? It is well known that the ARC-COLORING problem is NP-complete [14]. W.l.o.g., we assume that each edge of C_n is traversed by exactly k paths in F. If some edge $[i, i+1]$ of C_n is traversed by $r < k$ paths, then we can add $k-r$ paths of the form $< i, i+1 >$ (or $< i, 1 >$ if $i = n$) to F without changing its k-colorability. We assume that no path in F covers the cycle C_n entirely. Let I be an instance of the ARC-COLORING problem defined as above. We construct from I an instance I' of the permutation-path coloring problem on trees, consisting of a symmetric directed tree T, a collection of paths $\mathcal{P}$ on T and a positive integer k' such that I' verifies the constraints given in (a) (resp. (b), (c), (d)), and such that F is k-colorable if and only if $\mathcal{P}$ is k'-colorable.

Let $< i, j >$ be any path in F, thus we say that $< i, j >$ is of *type 1* (resp. *type 2*) if $i < j$ (resp. $i > j$). In Figure 2 we give an example of these two types of paths in F.

Proof of Part (a). T is constructed as follows. First, construct a graph on n vertices isomorphic to the path graph P_n and denote its vertices by $v_1, v_2, \ldots, v_n$. Next, construct $2(n+k)$ different isomorphic copies of the star graph ST(4). Take $n+k$ of these $2(n+k)$ isomorphic graphs and denote their leaves by l_i, s_i and t_i, and denote the leaves of the $n+k$ other ones by r_i, x_i and z_i, $1 \leq i \leq n+k$. Finally, connect vertex l_i (resp. r_i) to vertex l_{i+1} (resp. r_{i+1}), $1 \leq i \leq n+k-1$, and connect the vertex l_1 (resp. r_1) to vertex v_1 (resp. v_n) of P_n (see Figure 2).
$\mathcal{P}$ is constructed as follows (see Figure 2). For each path $< i, j > \in F$, if $< i, j >$ is of type 1 (i.e. $i < j$), then add to $\mathcal{P}$ the paths $A_{i,j} = v_i \rightsquigarrow v_j$ and $B_{j,i} = v_j \rightsquigarrow v_i$. Otherwise, if $< i, j >$ is of type 2 (i.e. $i > j$), then let p (resp. q) be an integer in $\{1, 2, \ldots, k\}$ such that no path in $\mathcal{P}$ uses the vertices x_p and z_p (resp. s_q and t_q) of T as ending vertices. Add to $\mathcal{P}$ the path sets $\bar{A}_{i,j} = \{v_i \rightsquigarrow z_p\ ,\ x_p \rightsquigarrow t_q\ ,\ s_q \rightsquigarrow v_j\}$ and $\bar{B}_{j,i} = \{v_j \rightsquigarrow s_q\ ,\ t_q \rightsquigarrow x_p\ ,\ z_p \rightsquigarrow v_i\}$.

In order to make sure that the (multi)-digraph associated to the tree and the collection of paths (see Def. 3) is connected (property that will be used to prove Part (c)), for each j, $k+1 \leq j \leq n+k$, we add to $\mathcal{P}$ the sets of paths $C_{j-k} = \{s_j \rightsquigarrow v_{j-k}, v_{j-k} \rightsquigarrow z_{j'}, x_{j'} \rightsquigarrow v_{j'-k}, v_{j'-k} \rightsquigarrow t_j\}$ and $D_{j-k} = \{t_j \rightsquigarrow v_{j'-k}, v_{j'-k} \rightsquigarrow x_{j'}, z_{j'} \rightsquigarrow v_{j-k}, v_{j-k} \rightsquigarrow s_j\}$, where $j' = j+1$ if $j < n+k$, otherwise $j' = k+1$.

In addition, for each i, $1 \leq i \leq n+k$, we add to $\mathcal{P}$ $2(n+k)-1$ identical paths $s_i \rightsquigarrow t_i$ (resp. $x_i \rightsquigarrow z_i$) and $2(n+k)-1$ identical paths $t_i \rightsquigarrow s_i$ (resp. $z_i \rightsquigarrow x_i$). Finally, set $k' = 2(n+k)$. In Figure 2 we present an example of this polynomial construction.

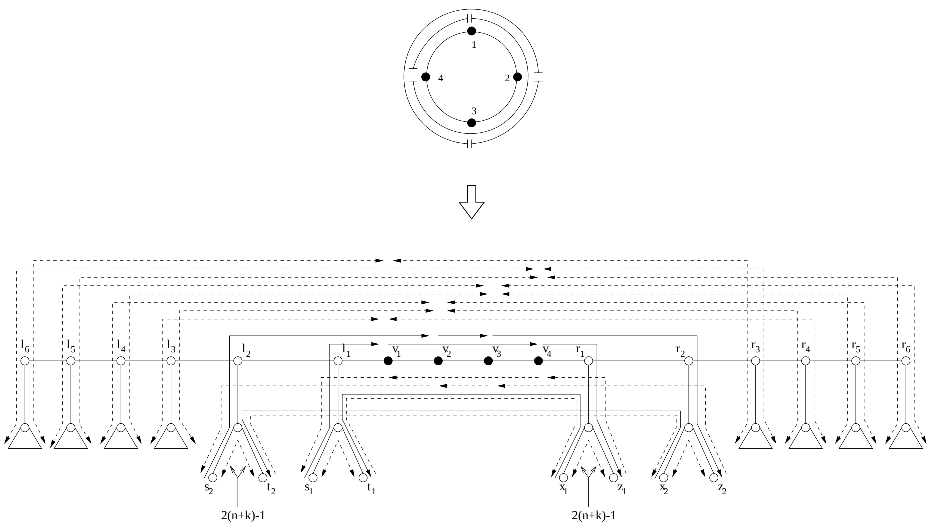
By construction, it is easy to see that T is a binary tree and $\mathcal{P}$ is a collection of symmetric paths on T. Moreover, let $< i_1, j_1 >, < i_2, j_2 >, \ldots, < i_k, j_k >$ be the k paths of type 2 in F, and let $\bar{A}_{i_r,j_r} = \{v_{i_r} \rightsquigarrow z_{p_r}\ ,\ x_{p_r} \rightsquigarrow t_{q_r}\ ,\ s_{q_r} \rightsquigarrow v_{j_r}\}$ and $\bar{B}_{j_r,i_r} = \{v_{j_r} \rightsquigarrow s_{q_r}\ ,\ t_{q_r} \rightsquigarrow x_{p_r}\ ,\ z_{p_r} \rightsquigarrow v_{i_r}\}$ be the two sets of paths in $\mathcal{P}$ associated with the path $< i_r, j_r >$, $1 \leq r \leq k$. Then $\mathcal{P}$ verifies the following properties.

Property 1 *All the paths in each of the sets* $\bar{A}_{i_r,j_r}$, $\bar{B}_{j_r,i_r}$, C_m, *and* D_m, $1 \le r \le k$, $1 \le m \le n$, *are colored with the same color in any* k'*-coloring of* $\mathcal{P}$.

Property 2 *Each of the sets* $\bar{A}_{i_r,j_r}$, $\bar{B}_{j_r,i_r}$, C_m, *and* D_m, $1 \le r \le k$, $1 \le m \le n$, *should be assigned a different color in any* k'*-coloring of* $\mathcal{P}$.

Property 3 *Each path* $A_{a,b}$ *(resp.* $B_{b,a}$*) in* $\mathcal{P}$ *associated with a path* $< a, b >$ *in* F *of type 1, intersects with all the paths in* $\cup_{r=1}^{k}\{t_{q_r} \rightsquigarrow x_{p_r} \in \bar{B}_{j_r,i_r}\}$ *(resp. in* $\cup_{r=1}^{k}\{x_{p_r} \rightsquigarrow t_{q_r} \in \bar{A}_{i_r,j_r}\}$*), and with at least one of the paths in each one of the sets* C_m *and* D_m, $1 \le m \le n$.

$F = \{< 1, 4 >, < 2, 3 >, < 4, 1 >, < 3, 2 >\}$, $n = 4$ and $k = 2$



$< 1, 4 > \longrightarrow A_{1,4} = v_1 \rightsquigarrow v_4$ and $B_{4,1} = v_4 \rightsquigarrow v_1$

$< 2, 3 > \longrightarrow A_{2,3} = v_2 \rightsquigarrow v_3$ and $B_{3,2} = v_3 \rightsquigarrow v_2$

$< 4, 1 > \longrightarrow \bar{A}_{4,1} = \{v_4 \rightsquigarrow z_1, x_1 \rightsquigarrow t_1, s_1 \rightsquigarrow v_1\}$ and $\bar{B}_{1,4} = \{v_1 \rightsquigarrow s_1, t_1 \rightsquigarrow x_1, z_1 \rightsquigarrow v_4\}$

$< 3, 2 > \longrightarrow \bar{A}_{3,2} = \{v_3 \rightsquigarrow z_2, x_2 \rightsquigarrow t_2, s_2 \rightsquigarrow v_2\}$ and $\bar{B}_{2,3} = \{v_2 \rightsquigarrow s_2, t_2 \rightsquigarrow x_2, z_2 \rightsquigarrow v_3\}$

$C_1 = \{s_3 \rightsquigarrow v_1, v_1 \rightsquigarrow z_4, x_4 \rightsquigarrow v_2, v_2 \rightsquigarrow t_3\}$ and $D_1 = \{t_3 \rightsquigarrow v_2, v_2 \rightsquigarrow x_4, z_4 \rightsquigarrow v_1, v_1 \rightsquigarrow s_3\}$

$C_2 = \{s_4 \rightsquigarrow v_2, v_2 \rightsquigarrow z_5, x_5 \rightsquigarrow v_3, v_3 \rightsquigarrow t_4\}$ and $D_2 = \{t_4 \rightsquigarrow v_3, v_3 \rightsquigarrow x_5, z_5 \rightsquigarrow v_2, v_2 \rightsquigarrow s_4\}$

$C_3 = \{s_5 \rightsquigarrow v_3, v_3 \rightsquigarrow z_6, x_6 \rightsquigarrow v_4, v_4 \rightsquigarrow t_5\}$ and $D_3 = \{t_5 \rightsquigarrow v_4, v_4 \rightsquigarrow x_6, z_6 \rightsquigarrow v_3, v_3 \rightsquigarrow s_5\}$

$C_4 = \{s_6 \rightsquigarrow v_4, v_4 \rightsquigarrow z_3, x_3 \rightsquigarrow v_1, v_1 \rightsquigarrow t_6\}$ and $D_4 = \{t_6 \rightsquigarrow v_1, v_1 \rightsquigarrow x_3, z_3 \rightsquigarrow v_4, v_4 \rightsquigarrow s_6\}$

Figure 2: Partial construction of I' from I.

For each j, $k+1 \leq j \leq n+k$, let C^g_{j-k} (resp. C^d_{j-k}) be the subset of C_{j-k} formed by paths $\{s_j \rightsquigarrow v_{j-k}, v_{j'-k} \rightsquigarrow t_j\}$ (resp. $\{v_{j-k} \rightsquigarrow z_{j'}, x_{j'} \rightsquigarrow v_{j'-k}\}$), where $j' = j+1$ if $j < n+k$, otherwise $j' = k+1$. In analogous way, let $D^g_{j-k} = \{t_j \rightsquigarrow v_{j'-k}, v_{j-k} \rightsquigarrow s_j\}$ and $D^d_{j-k} = \{v_{j'-k} \rightsquigarrow x_{j'}, z_{j'} \rightsquigarrow v_{j-k}\}$ be the subsets of D_{j-k}. On the one hand, by construction, the $k'-1$ identical paths $s_i \rightsquigarrow t_i$ (resp. $t_i \rightsquigarrow s_i$) and the $k'-1$ identical paths $x_i \rightsquigarrow z_i$ (resp. $z_i \rightsquigarrow x_i$), $1 \leq i \leq n+k$, make sure that all the paths in each one of the sets $\bar{A}_{i_r,j_r}$, $\bar{B}_{j_r,i_r}$, C^g_m, C^d_m, D^g_m and D^d_m, $1 \leq r \leq k$, $1 \leq m \leq n$, are colored with the same color in any k'-coloring of $\mathcal{P}$. On the other hand, it is easy to see that by construction, each path $x_{p_r} \rightsquigarrow t_{q_r} \in \bar{A}_{i_r,j_r}$ (resp. $t_{q_r} \rightsquigarrow x_{p_r} \in \bar{B}_{j_r,i_r}$), $1 \leq r \leq k$, intersects with all the paths in $\cup_{m=1}^{k}\{x_{p_m} \rightsquigarrow t_{q_m} \in \bar{A}_{i_m,j_m} : m \neq r\}$ (resp. in $\cup_{m=1}^{k}\{t_{q_m} \rightsquigarrow x_{p_m} \in \bar{B}_{j_m,i_m} : m \neq r\}$) and with all the paths in $\cup_{m=1}^{k}\{z_{p_m} \rightsquigarrow v_{i_m}, v_{j_m} \rightsquigarrow s_{q_m} \in \bar{B}_{j_m,i_m}\}$ (resp. in $\cup_{m=1}^{k}\{v_{i_m} \rightsquigarrow z_{p_m}, s_{q_m} \rightsquigarrow v_{j_m} \in \bar{A}_{i_m,j_m}\}$) (c.f. Figure 2). Indeed, each set of paths C_m (resp. D_m) intersects with all the paths in $\mathcal{P} \setminus C_m \setminus Q$ (resp. in $\mathcal{P} \setminus D_m \setminus Q$), where Q is the collection of all the $k'-1$ identical paths $s_i \rightsquigarrow t_i$ (resp. $t_i \rightsquigarrow s_i$) and the $k'-1$ identical paths $x_i \rightsquigarrow z_i$ (resp. $z_i \rightsquigarrow x_i$), $1 \leq i \leq k$ (c.f. Figure 2).

Suppose that $\mathcal{P}$ is k'-colorable and that there exists some set C_{j-k} such that the paths in their subsets C^g_{j-k} and C^d_{j-k} are colored using two different colors. By construction (i.e., the four ended vertices of the two paths in C^g_{j-k} (resp. in C^d_{j-k}) are pairwise different) and by previous remarks, any proper coloring of $\mathcal{P}$ needs at least $k'+1$ colors to be colored, which is a contradiction to the assumption that $\mathcal{P}$ is k'-colorable. Thus, all the paths in each one of sets C_m, $1 \leq m \leq n$, are colored with the same color in any k'-coloring of $\mathcal{P}$ (if there exists one). By a symmetric argument, the previous statement also holds for the paths in each one of the sets D_m. This proves the Properties 1 and 2. Finally, Property 3 follows directly from construction.

Now we claim that there is a k-coloring of F if and only if there is a k'-coloring of $\mathcal{P}$. Assume that there is a k-coloring of F, and let $< i,j >$ be any path in F colored with the color γ, $1 \leq \gamma \leq k$. Thus a k'-coloring of $\mathcal{P}$ can be carried out as follows: if $< i,j >$ is of type 1, then we color the paths $A_{i,j} = v_i \rightsquigarrow v_j$ and $B_{j,i} = v_j \rightsquigarrow v_i$ in $\mathcal{P}$ with colors γ and $\gamma + k$ respectively. Otherwise, if $< i,j >$ is of type 2, then we color all its three associated paths in $\bar{A}_{i,j}$ (resp. in $\bar{B}_{j,i}$) with color γ (resp. $\gamma + k$). Next, for each i, $1 \leq i \leq n$, we assign to all the paths in the set C_i (resp. D_i) the color $2k+i$ (resp. $2k+n+i$). Finally, for each i, $1 \leq i \leq n+k$, we color the $k'-1$ identical paths $s_i \rightsquigarrow t_i$ (resp. $x_i \rightsquigarrow z_i$) and the $k'-1$ identical paths $t_i \rightsquigarrow s_i$ (resp. $z_i \rightsquigarrow x_i$) with the $k'-1$ available colors for each one of these $(k'-1)$-sets of paths. Thus, by Properties P1, P2, and P3, it is easy to see that such a coloring is a proper k'-coloring of $\mathcal{P}$.

Conversely, assume that there is a k'-coloring of $\mathcal{P}$. By Properties P1, P2, and P3, it is easy to deduce two proper k-colorings for F as follows: if $< i,j >$ is a path in F of type

1, we assign to $< i, j >$ the color assigned to path $A_{i,j} = v_i \rightsquigarrow v_j$ (resp. $B_{j,i} = v_j \rightsquigarrow v_i$) in $\mathcal{P}$. Otherwise, if $< i, j >$ is a path in F of type 2, we assign to $< i, j >$ the same color assigned to the three paths in the set $\bar{A}_{i,j}$ (resp. $\bar{B}_{j,i}$). Thus, F is k-colorable if and only if $\mathcal{P}$ is k'-colorable which ends the proof of (a).

Proof of Part (b). This follows from (a). In fact, let T be the binary tree and $\mathcal{P}$ be the symmetric collection of paths constructed in Part (a). Let u and v be two adjacent vertices in T, and let $i(u, v)$ (resp. $o(u, v)$) be the subset of paths in $\mathcal{P}$ traversing the arc (u, v) (resp. (v, u)) and having as final-vertex (resp. initial-vertex) the vertex v. As $\mathcal{P}$ is symmetric, it is clear that $|i(u, v)| = |o(u, v)|$. Then, replace the pair of arcs (u, v) and (v, u) by a path graph on $|i(u, v)| = \alpha$ vertices. Let P_α denote such a path graph, and $w_1, w_2, \ldots, w_\alpha$ denote its vertices. Replace each pair of symmetric paths $a \rightsquigarrow v \in i(u, v)$ and $v \rightsquigarrow a \in o(u, v)$ by the paths $a \rightsquigarrow w_j$ and $w_j \rightsquigarrow a$, where w_j is a vertex of P_α not yet used by any path as initial or final vertex. Using the previous transformation on each pair of adjacent vertices of T, we obtain an instance consisting of an extended binary tree T' and a set of paths $\mathcal{P}$ which represents an involution of the vertices of T', which is equivalent (from the coloring point of view) to the one obtained in Part (a), ending the proof of (b).

Proof of Part (c). Let T be the binary tree and $\mathcal{P}$ be the symmetric collection of paths constructed in Part (a). Clearly, the digraph $\vec{G}_T(\mathcal{P})$ associated with $\mathcal{P}$ (see Def. 3) is a connected pseudo-symmetric digraph. First, we use a similar procedure as in Part (b) which maintains the connectedness of the digraph associated with $\mathcal{P}$ as follows: for each vertex v_i in T (recall that vertex v_i belongs to the initial path graph P_n constructed in Part (a)), $1 \leq i \leq n$, if u is an adjacent vertex to vertex v_i, and the pairs of arcs (u, v_i) and (v_i, u) should be replaced by a new path graph on α vertices denoted by $w_1, w_2, \ldots, w_\alpha$ (see Part (b)), where w_1 (resp. w_α) will be the new adjacent vertex to v_i (resp. u), then after this replacement we should add to $\mathcal{P}$ the paths $w_j \rightsquigarrow w_{j+1}$ and $w_{j+1} \rightsquigarrow w_j$, $1 \leq j < \alpha$, and the paths $w_1 \rightsquigarrow v_i$ and $v_i \rightsquigarrow w_1$. It is not difficult to see that this new instance is equivalent (from the coloring point of view) to the previous one, and that each inner vertex of the current tree is the initial or final vertex of at most three paths. Let T' and $\mathcal{P}'$ denote the current tree and the current symmetric collection of paths respectively, after the previous transformation. Then, the digraph $\vec{G}_{T'}(\mathcal{P}')$ is a connected pseudo-symmetric digraph, and by Theorem 1, $\vec{G}_{T'}(\mathcal{P}')$ is Eulerian and an Eulerian circuit can be found in polynomial-time. Let $a_1, a_2, \ldots, a_\rho, a_1$ be an Eulerian circuit of $\vec{G}_{T'}(\mathcal{P}')$, where $\rho = |\mathcal{P}'|$. Note that each pair (a_i, a_{i+1}) in the Eulerian circuit represents a path $a_i \rightsquigarrow a_{i+1}$ of $\mathcal{P}'$. Moreover, let $w_1, w_2, \ldots, w_{n'}$ denote the vertices of the current path graph in T', where w_1 is adjacent to vertex l_1, and vertex $w_{n'}$ is adjacent to vertex r_1. By the previous construction, each vertex w_i must be at most three times on the Eulerian circuit. Thus, following the Eulerian circuit in the order $a_1, a_2, \ldots, a_\rho, a_1$, for each vertex w_i, $1 \leq i \leq n'$, if w_i is found for the second or third time on the Eulerian circuit, we add a new vertex u_i to T' and connect it to vertex w_i, and we replace the paths $\beta \rightsquigarrow w_i$ and $w_i \rightsquigarrow \gamma$ in $\mathcal{P}'$, by the paths $\beta \rightsquigarrow u_i$ and $u_i \rightsquigarrow \gamma$, where β and γ are the immediate predecessor and successor of w_i respectively, on the Eulerian circuit. Indeed, by construction, each one of the vertices s_i, t_i, x_i, and z_i, $1 \leq i \leq n+k$, is found exactly k' times on the Eulerian circuit, and given that each one of these vertices is a leaf of T', we can replace each one of these vertices by a new path on k' vertices and arrange the k' paths in $\mathcal{P}'$ ending and beginning in each one of these in agreement with the Eulerian circuit. Therefore, it is easy to prove that the obtained tree T' has maximum degree at most equal to 4 and that the set of paths $\mathcal{P}'$ represents a circular permutation of the vertices of T'. Thus, taking care of the initial paths $A_{i,j}$ and $B_{j,i}$ (resp. set of paths $\bar{A}_{i,j}$ and $\bar{B}_{j,i}$) associated with paths $< i,j >$ in F of type 1 (resp. 2), we obtain that the final circular permutation set of paths $\mathcal{P}'$ on T' is k'-colorable if and only if F is k-colorable, which ends the proof of (c).

Proof of Part (d). This follows from (a) and (b). In fact, let T be the binary tree and $\mathcal{P}$ be the symmetric collection of paths constructed in Part (a). Replace all the $n+k$ isomorphic star subgraphs on 4 vertices, where the i^{th} one has its leaves labeled by l_i, s_i, and t_i (resp. r_i, x_i, and z_i) by only one star $\mathrm{ST}(2(n+k)+1)$ having as its leaves the vertices s_i and t_i (resp. x_i and z_i), $1 \leq i \leq n+k$, and denote by l_1 (resp. r_1) its only vertex of degree $2(n+k)$. Next, connect the vertex l_1 (resp. r_1) to vertex v_1 (resp. v_n) of P_n, leaving $\mathcal{P}$ as in (a). Thus, it is easy to see that this new instance is equivalent to the one obtained in Part (a) (from the coloring point of view). Finally, using similar arguments as in Part (b), we prove the NP-hardness for the involution case. This ends the proof of (d) and the theorem is proved. □

By Proposition 1 (resp. Proposition 2), the best known approximation algorithm for coloring any collection of paths (resp. symmetric paths) with load L on any tree network uses at most $\lceil \frac{5}{3}L \rceil$ (resp. $\lfloor \frac{3}{2}L \rfloor$) colors. Therefore it trivially also holds for any permutation-set (resp. involution-set) of paths with load L on any tree.

3.2 Some polynomial cases

Let $\mathcal{P}$ be any collection of paths on a tree network T. If T is a linear network, then the minimum number of colors $R_T(\mathcal{P})$ needed to color the paths in $\mathcal{P}$ is equal to the load $L_T(\mathcal{P})$ induced by $\mathcal{P}$. In fact, if T is a linear network, then the conflict graph of the paths in $\mathcal{P}$ is an *interval graph* (see [16]). Moreover, optimal vertex coloring for interval graphs can be computed efficiently [18]. When T is a star network, the equality between $R_T(\mathcal{P})$ and $L_T(\mathcal{P})$ also holds because the path coloring problem on these graphs is equivalent to finding a minimum edge-coloring of an undirected bipartite graph. Moreover, the minimum number of colors needed to color the edges of a bipartite graph is equal to its maximum degree, and such an edge-coloring in these graphs can be found in polynomial time [3]. Combining these approaches for linear and star networks, Gargano et al. [15] show that if T is a generalized

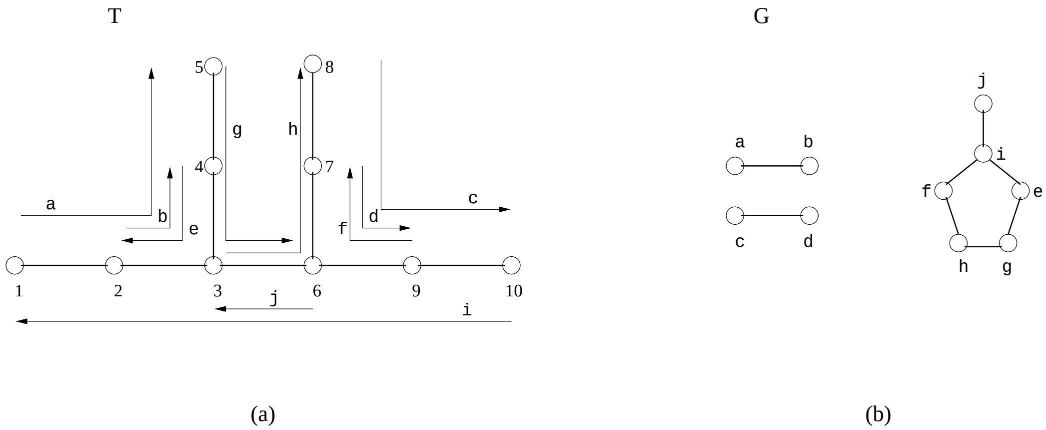


Figure 3: (a) A tree T on 10 vertices and a permutation $\sigma = (5, 4, 8, 2, 6, 3, 9, 10, 7, 1)$ to be routed on T. (b) The conflict graph G associated with permutation σ in (a).

star network then, an optimal coloring of $\mathcal{P}$ on these networks can be computed efficiently in polynomial time, and that the equality between $R_T(\mathcal{P})$ and $L_T(\mathcal{P})$ also holds. Note that all the results in these three networks trivially hold when $\mathcal{P}$ is a permutation-set of paths. However, by Theorem 2, it suffices that the tree network T has two vertices with degree greater than two and the permutation-path coloring problem on these networks becomes NP-hard. Moreover, in binary tree networks having only two vertices with degree equal to 3, the equality between the load and the minimum number of colors for a permutation-path set does not always hold as we can see in Figure 3. In fact, Figure 3(a) shows an example of a permutation $\sigma \in \mathcal{S}_{10}$ to be routed on a tree T on 10 vertices, whose load $L_T(\sigma)$ is equal to 2. Figure 3(b) shows the conflict graph $G = G_T(\sigma)$. Thus, clearly $R_T(\sigma)$ is equal to the chromatic number of G. Therefore, as the conflict graph G has the cycle C_5 as induced subgraph, then the chromatic number of G is equal to 3, and thus $R_T(\sigma) = 3$.

All results obtained at present are worst-case complexity ones. However, it will be interesting from the practical point of view to compute the average coloring number of permutations to be routed on networks. In the following sections we study the average complexity of the coloring number of permutations. We begin our study in Section 4 by giving a lower bound for the average coloring number of permutations to be routed on arbitrary topology networks. In the remaining sections, we concentrate our study on the average coloring number of permutations on tree networks.

4 A lower bound for the average coloring number

We derive a lower bound for the average coloring number of permutations to be routed on arbitrary networks, by giving a lower bound for the average load of permutations to be routed. Let $G = (V, A)$ be a directed symmetric graph on n vertices (i.e. $|V| = n$) and r a routing function in G which assigns a set of paths on G to route any permutation $\sigma \in \mathcal{S}_n$. Let $\overline{L}_{G,r}$ be the average load of all permutations in $\mathcal{S}_n$ induced by the routing function r, and let $U \subseteq V$ be a subset of the vertex set of G. We denote by $c(U)$ the cut $(U, \bar{U})$, i.e., the set of arcs $\{(u, v) \in A : \ u \in U, \ v \in V \setminus U\}$.

Proposition 3 *For any graph $G = (V, A)$ on n vertices, and any routing function r in G,*

$$\overline{L}_{G,r} \geq \frac{1}{n} \cdot \max_{U \subseteq V} \left(\frac{|U| \cdot (n - |U|)}{|c(U)|} \right) \ .$$

Proof. Let $U \subseteq V$ be any subset of vertices in G and consider a permutation $\sigma \in \mathcal{S}_n$ to be routed on G by using the routing function r. The load of all the arcs in $c(U)$ induced by σ with the routing function r, that we denote by $L_r(U, \sigma)$, is at least equal to $|\{j \in U : \ \sigma(j) \notin U\}|$. Thus, the global load of $c(U)$, that we denote by $L_r(U)$, is at least equal to $\sum_{\sigma \in \mathcal{S}_n} L_r(U, \sigma)$. In fact, for any vertex $j \in U$ and for any vertex $k \in V \setminus U$, each permutation $\sigma \in \mathcal{S}_n$ such that $\sigma(j) = k$ contributes for at least one unit to the global load of $c(U)$. Therefore the average load of $c(U)$ verifies $\overline{L}_r(U) \geq \frac{1}{n!} \sum_{j \in U} \sum_{k \in V \setminus U} |\{\sigma \in \mathcal{S}_n : \ \sigma(j) = k\}|$. Moreover, for all pairs of vertices j and k in G, there exist $(n-1)!$ permutations $\sigma \in \mathcal{S}_n$ such that $\sigma(j) = k$. Therefore, $\overline{L}_r(U) \geq \frac{1}{n!} \sum_{j \in U} \sum_{k \in V \setminus U} (n-1)! = \frac{1}{n} \sum_{j \in U} \sum_{k \in V \setminus U} 1 = \frac{|U|(n-|U|)}{n}$. Thus, for some arc $\alpha \in c(U)$, the average load of α verifies $\overline{L}_r(\alpha) \geq \frac{\overline{L}_r(U)}{|c(U)|}$. So, $\overline{L}_{G,r} \geq \frac{1}{n} \cdot \max_{U \subseteq V} \left(\frac{|U| \cdot (n-|U|)}{|c(U)|} \right)$. □

Let us denote by $\mathcal{C}(G)$ the parameter $\max_{U \subseteq V} \left(\frac{|U| \cdot (|V| - |U|)}{|c(U)|} \right)$. It is not difficult to see that $\mathcal{C}(G)$ is equal to $\frac{1}{\mathcal{S}(G)}$, where $\mathcal{S}(G)$ denotes the *sparsest cut* of graph G. In fact, $\mathcal{S}(G)$ is defined by $\min_{U \subseteq V} \left(\frac{|c(U)|}{|U| \cdot (|V| - |U|)} \right)$. In [23] it is shown that computing the sparsest cut of a graph is NP-hard, which implies that computing the parameter $\mathcal{C}(G)$ is also NP-hard, and so computing the best lower bound given in Proposition 3 is NP-hard. Moreover, for any constant k, if the *edge-bisection* of G, i.e., a cut $c(U)$ of G with minimal cardinality and such that $|U| = \lfloor \frac{n}{2} \rfloor$, is at most equal to k, then the parameter $\mathcal{C}(G)$ can be computed in polynomial-time. In fact, it is clear that $\mathcal{C}(G) \geq \frac{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}{k}$. Consider that there exists $V_1 \subset V(G)$ such that $\frac{|V_1|(n-|V_1|)}{|c(V_1)|} > \frac{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}{k}$. Thus, since $|V_1|(n - |V_1|) \leq \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$, we have $|c(V_1)| < k$. So, to compute $\mathcal{C}(G)$ it is enough to consider all the subsets of $A(G)$ disconnecting G with maximal cardinality at most equal to $k - 1$, i.e., a polynomial number of such subsets. For example, for any $2d$-mesh $M(2n, k)$ with $2n$ lines and a constant number k of columns, $\mathcal{C}(M(2n, k)) = k \cdot n^2$, and for any ring C_n, $\mathcal{C}(C_n) = \frac{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}{2}$.

We conclude this section by giving a lower bound on the average number of colors needed to color any permutation $\sigma \in \mathcal{S}_n$ on any tree on n vertices as follows.

Let T be a tree on n vertices. By Proposition 3, we can deduce that the average load of any arc i^+ of T, $1 \leq i \leq n-1$, denoted by $\bar{L}_T(i)$, verifies $\bar{L}_T(i) = \dfrac{|T(i)|(n - |T(i)|)}{n}$. Moreover, for any vertex i of T, let $v_T(i) = |T(i)|/n$ and $\tilde{v}_T(i) = \min(v_T(i), 1 - v_T(i))$. Let $\tilde{v}_T = \max_i \tilde{v}_T(i)$. Then, it is clear that $\max_i\{\bar{L}_T(i)\} = n\tilde{v}_T(1 - \tilde{v}_T)$. Indeed, it is straightforward that $\bar{L}_T \geq \max_i\{\bar{L}_T(i)\}$. Therefore, we obtain a lower bound for the average load $\bar{L}_T$.

Lemma 4 *$\bar{L}_T \geq n\tilde{v}_T(1 - \tilde{v}_T)$.*

Moreover, as $\bar{R}_T \geq \bar{L}_T$, we obtain the following lower bound on the average number of colors needed to color any permutation $\sigma \in \mathcal{S}_n$ on any tree T on n vertices.

Lemma 5 *$\bar{R}_T \geq n\tilde{v}_T(1 - \tilde{v}_T)$.*

5 Average coloring number on linear networks

The main result of this section is the following:

Theorem 3 *The average coloring number of the permutations in $\mathcal{S}_n$ to be routed on a linear network on n vertices is*

$$\frac{n}{4} + \frac{\lambda}{2} n^{1/3} + O(n^{1/6}), \quad n \to \infty$$

where $\lambda = 0.99615\ldots$.

To prove this result, we use enumerative and asymptotic combinatorial techniques. Our approach first uses the same methodology as Lagarias et al. [20] who studied involutions with no fixed point routed on the linear network. At first we recall in Subsection 5.1 a bijection between permutations in $\mathcal{S}_n$ and special walks in $\mathbb{N} \times \mathbb{N}$, called "Motzkin walks" [2]. The bijection is such that the height of the walk is equal to the load of the permutation. We get in Subsection 5.2 the generating function of permutations with coloring number k, for any given k. This gives rise to an algorithm to compute exactly the average coloring number of the permutations for any fixed n. Then we are able to combine these enumerative results with random walks techniques developped by Louchard [22] and Daniels and Skyrme [7] to prove Theorem 3. Note that this "random walk" approach was not developed in [20] and we therefore extend our results for permutations to involutions with no fixed point in Section 7.

5.1 A bijection between permutations and Motzkin walks

A **Motzkin walk** on $\mathbb{N} \times \mathbb{N}$ of length n is an n-tuple $w = (w_1, w_2, \ldots, w_n)$ of unitary steps (North-East, South-East or East). Let h_i be the height of the i^{th} step that is the difference between the number of North-East and South-East steps in $(w_1, w_2, \ldots, w_i)$, $1 \leq i \leq n$. Then the walk must satisfy the following conditions:

- $h_i \geq 0$, $1 \leq i \leq n$;
- $h_n = 0$;

The **height** of a Motzkin walk w is $H(w) = \max_{i \in \{0,1,\ldots,n\}} \{h_i\}$.

Given two infinite sequences $\{\lambda_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 0}$, a **labeled Motzkin walk** of length n has the shape of a Motzkin walk and the South-East steps starting at height i can be labeled from 1 to λ_i and the East steps of height i can be labeled from 1 to b_i. Moreover, given two sequences $\{\lambda_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 0}$, let $\mathcal{M}_n$ be the number of labeled Motzkin walks and $\mathcal{M}(z) = \sum_{n\geq 0} \mathcal{M}_n z^n$ the associated generating function.

Proposition 4 *[10, 27] The generating function $\mathcal{M}(z)$ is a continued fraction. Its expression is*

$$\mathcal{M}(z) = \cfrac{1}{1 - b_0 z - \cfrac{\lambda_1 z^2}{1 - b_1 z - \cfrac{\lambda_2 z^2}{1 - b_2 z - \cfrac{\lambda_3 z^2}{1 - b_3 z - \cfrac{\lambda_4 z^2}{\vdots}}}}}.$$

Labeled Motzkin walks are in relation with several well-studied combinatorial objects [10, 27] and in particular with permutations. The walks we will deal with are labeled as follows:

- each South-East step starting at height i is labeled by an integer between 1 and i^2 (or, equivalently, by a pair of integers, each one between 1 and i);
- each East step of height i is labeled by an integer between 1 and $2i + 1$.

In Figure 4 we present an example of the labeled Motzkin walks we consider.

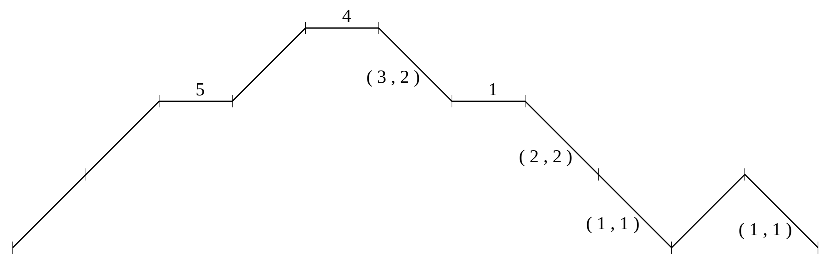


Figure 4: Example of a labeled Motzkin walk of length 11 and height 3.

Let W_n be the set of such labeled Motzkin walks of length n. We recall that $\mathcal{S}_n$ is the set of permutations on $[n]$. The following result was first established by Françon and Viennot [12]:

Theorem (Françon-Viennot) *There is a one-to-one correspondence between the elements of W_n and the elements of $\mathcal{S}_n$.*

Several bijective proofs of this theorem are known. Biane's bijection [2] is particular, in the sense that it preserves the height: to any labeled Motzkin walk of length n and height k corresponds a permutation in $\mathcal{S}_n$ with load k (and so with coloring number k). We present in what follows another version of Biane's bijection in order to understand the relationship between the height of the labeled Motzkin walks and the load of the permutations.

The bijection We will explain the mapping ϕ from the labeled Motzkin walks of length n to the permutations in $\mathcal{S}_n$ on the linear network on n vertices. The reverse is easy and left to the reader, see [2] for more details. Consider a linear network P_n on n vertices such that the vertices are labeled from left to right from 1 to n. Thus, Biane's correspondence between a labeled Motzkin walk $w = (w_1, w_2, \ldots, w_n)$ and a permutation $\phi(w) = \sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ on P_n is such that, for $1 \leq i \leq n$:

- w_i is an East step of height j and is labeled $2j+1$ if and only if $\sigma(i) = i$.
- w_i is an East step of height j and is labeled l with $1 \leq l \leq j$ if and only if $\sigma^{-1}(i) < i$ and $\sigma(i) > i$.
- w_i is an East step of height j and is labeled l with $j+1 \leq l \leq 2j$ if and only if $\sigma^{-1}(i) > i$ and $\sigma(i) < i$.
- w_i is a North-East step if and only if $\sigma(i) > i$ and $\sigma^{-1}(i) > i$.
- w_i is a South-East step if and only if $\sigma(i) < i$ and $\sigma^{-1}(i) < i$.

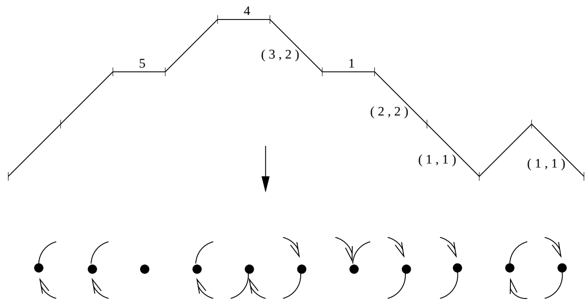


Figure 5: From the shape of the path to the shape of the permutation

The previous correspondence automatically gives us that the height of each step w_i is equal to $L_{P_n}(\sigma, i^+)$ for $1 \leq i \leq n-1$. In fact, by previous correspondence, we have that $L_{P_n}(\sigma, i^+)$ is equal to the number of integers $j \leq i$ such that $\sigma(j) > j$ and $\sigma^{-1}(j) > j$, minus the number of integers $j \leq i$ such that $\sigma(j) < j$ and $\sigma^{-1}(j) < j$, which corresponds exactly with the Equation (1) obtained in the proof of Lemma 1 (see Section 2).

Given a labeled Motzkin walk, it is easy to draw the shape of the permutation σ (beginning and end of the path $i \rightsquigarrow \sigma(i)$, $1 \leq i \leq n$), using the previous correspondence. The beginning of the path $i \rightsquigarrow \sigma(i)$ uses arc i^+ in P_n if and only if w_i is a North East step or an East step at height j with a label between $j+1$ and $2j$. The beginning of the path $i \rightsquigarrow \sigma(i)$ uses arc $(i-1)^-$ in P_n if and only if w_i is a South East step or an East step at height j with a label between 1 and j. The end of the path $\sigma^{-1}(i) \rightsquigarrow i$ uses arc i^- in P_n if and only if w_i is a North East step or an East step at height j with a label between 1 and j. The end of the path $\sigma^{-1}(i) \rightsquigarrow i$ uses arc $(i-1)^+$ in P_n if and only if w_i is a South East step or an East step at height j with a label between $j+1$ and $2j$. An example is illustrated in Figure 5. Now we label the shape of the permutation to keep all the information of the labeled Motzkin path. For i from 2 to n, if w_i is a South-East step with label (x, y) then we label the end of the path $\sigma^{-1}(i) \rightsquigarrow i$ by x and the beginning of the path $i \rightsquigarrow \sigma(i)$ by y. For i from 2 to n, if w_i is an East step of height j labeled by l; if $j+1 \leq l \leq 2j$ then we label the beginning of the path $i \rightsquigarrow \sigma(i)$ by $l-j$; if $1 \leq l \leq j$ then we label the end of the path $\sigma^{-1}(i) \rightsquigarrow i$ by l. See Figure 6 for an example of labeling. Finally we associate free beginnings and ends of paths going from left to right. For any free end of a path (resp. beginning of the path) labeled x, we associate with it the x^{th} free unlabeled beginning of a path (resp. unlabeled end of a path) starting from the left. See Figure 6 for an example of the construction of the permutation. $\square$

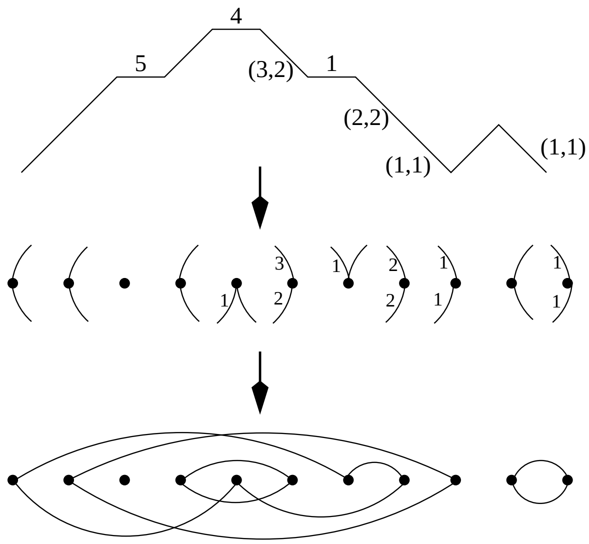


Figure 6: From a labeled Motzkin walk to a permutation

5.2 An algorithm to compute exactly the average coloring number

From Biane's bijection and Proposition 4, we can directly get the generating function of the permutations in $\mathcal{S}_n$, of coloring number at most k, to be routed on a linear network on n vertices,

$$H_{\leq k}(z) = \sum_{n\geq 0} \sum_{\sigma\in\mathcal{S}_{n,\leq k}} z^n$$

that is

$$\cfrac{1}{1 - z - \cfrac{z^2}{1 - 3z - \cfrac{4z^2}{1 - 5z - \cfrac{\vdots}{1-(2k-1)z - \cfrac{k^2z^2}{1-(2k+1)z}}}}}$$

Note that this generating function is rational for any fixed k. We can also use known results in enumerative combinatorics [10, 27] to get the generating function of the permutations of coloring number **exactly** k,

$$H_k(z) = \sum_{n\geq 0} \sum_{\sigma\in\mathcal{S}_{n,k}} z^n$$

n	$\bar{h}(n)$	n	$\bar{h}(n)$	n	$\bar{h}(n)$	n	$\bar{h}(n)$	n	$\bar{h}(n)$
1	0	9	2.60	17	4.83	25	7.00	33	9.13
2	0.5	10	2.88	18	5.10	26	7.27	34	9.40
3	0.83	11	3.16	19	5.37	27	7.53	35	9.66
4	1.12	12	3.44	20	5.65	28	7.80	36	9.93
5	1.43	13	3.72	21	5.92	29	8.07	37	10.19
6	1.73	14	4.00	22	6.19	30	8.33	38	10.46
7	2.02	15	4.27	23	6.46	31	8.60	39	10.72
8	2.31	16	4.55	24	6.73	32	8.87	40	10.99

Table 1: Average coloring number of permutations in $\mathcal{S}_n$.

that is

$$\frac{(k!)^2 z^{2k}}{P^*_{k+1}(z)P^*_k(z)}$$

with $P_0(z) = 1$, $P_1(z) = z - 1$ and $P_{n+1}(z) = (z - 2n - 1)P_n(z) - n^2 P_{n-1}(z)$ for $n \geq 1$, where P^* is the reciprocal polynomial of P, that is $P^*_n(z) = z^n P_n(1/z)$ for $n \geq 0$.

This generating function leads to a recursive algorithm to compute the number $h_{n,k}$ of permutations with coloring number k.

Proposition 5 *The number of permutations in $\mathcal{S}_n$ to be routed on a linear network on n vertices with coloring number k, satisfies the following recurrence*

$$h_{n,k} = \begin{cases} 0 & \text{if } n < 2k \\ (k!)^2 & \text{if } n = 2k \\ -\sum_{i=1}^{2k+1} p_i h_{n-i,k} & \text{otherwise} \end{cases}$$

where p_i is the coefficient of z^i in $P^*_{k+1}(z)P^*_k(z)$.

From this result we are able to compute the average coloring number $\bar{h}(n)$ of permutations in $\mathcal{S}_n$ to be routed on a linear network on n vertices, that is $\sum_{k\geq 0} kh_{n,k}/n!$. The first values are presented in Table 1.

5.3 Proof of Theorem 3

In [22], Louchard analyzes some list structures; in particular his "dictionary structure" corresponds to our labeled Motzkin walks. We will use his notation in order to refer directly to his article. All the results of this section are asymptotic. Restating Louchard's Theorem 6.2, we get the following lemma:

Lemma 6 *Let P_n be the path graph on n vertices and let σ^* be a permutation in $\mathcal{S}_n$. The load $L_{P_n}(\sigma^*, \lfloor nv \rfloor)$ of any arc $\lfloor nv \rfloor$ of P_n ($v \in [0,1]$) induced by σ^* has the following behavior*

$$L_{P_n}(\sigma^*, \lfloor nv \rfloor) = nv(1-v) + X(v)\sqrt{n} + O(1), \qquad n \to \infty$$

where X is a Markovian process with mean 0 and covariance $C(s,t) = 2s^2(1-t)^2$, $s \leq t$.

The works of Daniels and Skyrme [7] and Daniels [6] give us a way to compute the maximum of $L_{P_n}(\sigma^*, \lfloor nv \rfloor)$, that is, the load of a random permutation. These results have been applied in [13]. Let us now present them.

Let $X(v)$ be a Gaussian process with mean 0 and covariance $C(s,t)$ superposed on a curve $\tilde{y}(v)$. Assume that $\tilde{y}(v)$ is given by $\sqrt{n}y(v)$, $n \gg 1$ and that it has a unique maximum at $\bar{v}$. It is equivalent to look for its maximum $m = \max[X(v) + \tilde{y}(v)]$ and the time v^* at which this maximum occurs, or to search for the hitting time of $X(v)$ to the absorbing boundary. Daniels and Skyrme [7] have computed the asymptotic hitting time and place density. In the gaussian process case with covariance $C(s,t)$, $s \leq t$, Daniels [6] has matched the local behavior of $C(s,t)$ with the Brownian motion (or one of its variants) covariance near $\bar{v}$. We can deduce the density of the maximum m and time v^* from Equation (3.8) of [6] and Equation (5.9) of [6]. We first need to introduce some notations :

$$c_1 = \frac{\partial C}{\partial s}(\bar{v}, \bar{v}), \quad c_2 = \frac{\partial C}{\partial t}(\bar{v}, \bar{v}), \quad c = C(\bar{v}, \bar{v}) \tag{2}$$

$$v_0 = \bar{v} - c/c_1, \quad v_0 + V = \bar{v} + c/|c_2|, \quad V = cA/(c_1|c_2|) \tag{3}$$

$$A = c_1 + |c_2|, \quad B = -y''(\bar{v}), \quad u = n^{1/3}A^{-1/3}B^{2/3}(v^* - \bar{v}) \tag{4}$$

Let also $R(x) = \exp(x^3/6)H(x)$ with :

$$H(x) = 2^{-1/3}\frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{sx}\frac{ds}{A_i(2^{1/3}s)},$$

A_i is the classical Airy function. Let $f(x) = 2R(x)R(-x)$ and $v(x) = H'(x)/H(x)$; R and v are tabuled in Daniels [6]. Note that $f'(0) = 0$. Finally define :

$$\lambda = \int_{-\infty}^{+\infty} [R(x) - x^+]dx = 0.99615\ldots,$$

The results of Daniels and Skyrme are :

Theorem 4 *The random variable $m = max_v(\tilde{y}(v) + X(v))$ is asymptotically Gaussian with mean and variance*

$$E(m) \sim \lambda n^{-1/6} A^{2/3} B^{-1/3}, \qquad \sigma^2(m) \sim c. \tag{5}$$

The conditioned maximum $m|v^*$ is asymptotically gaussian with mean and variance

$$E(m|v^*) \sim \lambda n^{-1/6} A^{-1/3} [c_1 v(-u) + |c_2| v(u)] B^{1/3}, \qquad \sigma^2(m|v^*) \sim c. \tag{6}$$

The joint density of m and v^* is given by

$$\phi(m,u)\mathrm{d}m\ \mathrm{d}u = 2\sqrt{\frac{1}{2\pi c}} e^{-m^2/(2c)} \left(\frac{f(u)}{2} + n^{-1/6} A^{-1/3} B^{-1/3} m \phi_1(m,u) + O(n^{-1/3}) \right) \mathrm{d}m\ \mathrm{d}u \tag{7}$$

with

$$\phi_1(m,u) = -\frac{1}{4} u^2 f(u) \frac{A}{c} + R'(u) R(u) \frac{c_1}{c} + R(-u) R'(u) \frac{|c_2|}{c}$$

and where u has density $f(u)$. All expectations and densities have relative errors of order $O(n^{-1/3})$.

Let us now apply this theorem to our purpose. From Louchard's result we know that

$$L_{P_n}(\sigma^*, \lfloor nv \rfloor) = \sqrt{n}\left(\sqrt{n} v(1-v) + X(v)\right) + O(1).$$

Therefore we have $y(v) = v(1-v)$ and the unique maximum is attained at $\bar{v} = 1/2$. The covariance of our Gaussian process is $C(s,t) = 2s^2(1-t)^2$. We then obtain :

$$c = 1/8, \quad c_1 = 1/2, \quad c_2 = -1/2, \quad B = 2, \quad A = 1. \tag{8}$$

We can state now our result.

Proposition 6 *The coloring number of a random permutation $\sigma^* \in \mathcal{S}_n$ routed on P_n is :*

$$\max_v L_{P_n}(\sigma^*, \lfloor nv \rfloor) = \frac{n}{4} + m\sqrt{n} + O(n^{1/6}), \tag{9}$$

where m is asymptotically Gaussian with mean $E(m) \sim \lambda n^{-1/6}/2$ and variance $\sigma^2(m) \sim 1/8$. The variable m is characterized by (2)-(7). The arc $\lfloor nv^* \rfloor$ where the maximum occurs is a random variable characterized by (6) and (7). The constants are given in (8).

In the Equation (9) of Proposition 6, the only non-deterministic part is m which is Gaussian. So we just have to replace m by $E(m)$ to get the average of the coloring number and hence to prove Theorem 3, which states that the average of the coloring number of the permutations in $\mathcal{S}_n$ to be routed on a linear network on n vertices is $n/4 + \Theta(n^{1/3})$ when n goes to infinity. We can also get directly the variance.

Theorem 5 *The variance of the coloring number of the permutations in $\mathcal{S}_n$ to be routed on a linear network on n vertices is $\frac{n}{8}$.*

Proof. The variance of $\max_v L_{P_n}(\sigma^*, \lfloor nv \rfloor)$ is just $n\sigma^2(m)$. $\square$

6 Average coloring number on arbitrary tree networks

In this section, we extend the average complexity results on linear networks obtained in Section 4 to arbitrary tree networks. Given a tree T on n vertices, by Theorem 2, we know that it is NP-hard to compute $R_T(\sigma)$ for a permutation σ even if T is a binary tree and σ is an involution. By Proposition 1, we know that computing $R_T(\sigma)$ is $5/3-$approximable. The aim of this section is to find the average coloring number required for this approximation algorithm.

By Lemma 2, we know that $\bar{L}_T \leq \bar{R}_T \leq \frac{5}{3}\bar{L}_T + 1$. Therefore, we will compute the average load $\bar{L}_T$ for any tree T and will obtain bounds on $\bar{R}_T$, the average number of colors needed to color any permutation-path set on T. In Section 6.1 we present an upper bound for the average coloring number on tree networks. In Section 6.2 we obtain exact results on the average number of colors needed to color any permutation-path set on generalized star tree networks.

6.1 Upper bound

Let us remark that for any tree T on n vertices and for each vertex i of T, there exists a relabeling of the vertex set of T such that, for any permutation $\sigma \in \mathcal{S}_n$, $L_T(\sigma, i^+) = L_{P_n}(\sigma, |T(i)|)$. The vertices of $T(i)$ are relabeled with integers in $\{1, 2, \ldots, |T(i)|\}$, and the vertices in $T \setminus T(i)$ are relabeled with integers in $\{|T(i)|+1, \ldots, n\}$. Therefore, Lemma 6 can be rewritten as follows.

Lemma 7 *Let T be a tree on n vertices and let σ^* be a random permutation in $\mathcal{S}_n$. The load of any arc i^+ of T induced by σ^*, denoted by $L_T(\sigma^*, i^+)$, has the following behavior*

$$L_T(\sigma^*, i^+) = nv_T(i)(1 - v_T(i)) + X(v_T(i))\sqrt{n} + O(1),$$

where X is a Markovian process with mean 0 and covariance $C(s,t) = 2s^2(1-t)^2$, $s \leq t$, and where $v_T(i) = |T(i)|/n$.

As X is a Gaussian process with mean 0 and covariance $C(s,t) = 2s^2(1-t)^2$, we get easily that $L_T(\sigma^*, i^+) = nv_T(i)(1 - v_T(i)) + O(\sqrt{n})$ for any random permutation σ^*. This means that $L_T(\sigma^*) = n\tilde{v}_T(1 - \tilde{v}_T) + O(\sqrt{n})$ for any random permutation σ^*. As defined before, for any vertex i of T, $\tilde{v}_T(i) = \min(v_T(i), 1 - v_T(i))$ and $\tilde{v}_T = \max_i \tilde{v}_T(i)$. Thus, we obtain the following theorem :

Theorem 6 *The average load induced by all permutations $\sigma \in \mathcal{S}_n$ on T is*

$$\bar{L}_T = n\tilde{v}_T(1 - \tilde{v}_T) + O(\sqrt{n}).$$

From Lemma 2 and Theorem 6, we obtain the following upper bound on the average number of colors needed to color any permutation $\sigma \in \mathcal{S}_n$ on any tree T on n vertices.

Theorem 7 *For all ϵ, there exists $n_0 = n_0(\epsilon)$ such that, for all $n \geq n_0$ and any tree T on n vertices, the average number of colors $\bar{R}_T$ needed to color any permutation $\sigma \in \mathcal{S}_n$ on T verifies $\bar{R}_T \leq \left(\frac{5}{3} + \epsilon\right) n\tilde{v}_T(1 - \tilde{v}_T)$.*

6.2 The average number of colors in generalized star trees

Let k be a fixed integer, λ be a partition of $n-1$ in k parts and $\mathrm{GST}(\lambda)$ be the associated generalized star tree on n vertices. In this case, we have $\tilde{v}_{\mathrm{GST}(\lambda)} = \min(\lfloor n/2 \rfloor, \lambda_1)$. Moreover, Gargano et al. have shown in [15] that for any collection of paths $\mathcal{P}$ on a generalized star $\mathrm{GST}(\lambda)$, $R_{\mathrm{GST}(\lambda)}(\mathcal{P}) = L_{\mathrm{GST}(\lambda)}(\mathcal{P})$. Therefore, we can now apply the results of the previous subsection to get :

Theorem 8 *The average number of colors needed to color any permutation $\sigma \in \mathcal{S}_n$ on a generalized star tree $GST(\lambda)$ having n vertices is*

$$\bar{R}_{GST(\lambda)} = n\tilde{v}_{GST(\lambda)}(1 - \tilde{v}_{GST(\lambda)}) + O(\sqrt{n}).$$

In particular we obtain the following result. Let k be a fixed integer greater than 2.

Theorem 9 *The average number of colors needed to color any permutation $\sigma \in \mathcal{S}_{nk+1}$ on a generalized star tree $GST(\lambda)$ having $nk+1$ vertices and k branches of length n is $n(k-1)/k + O(\sqrt{n})$.*

7 Average coloring number for involutions

Given a tree T on $2n$ vertices, by Theorem 2, we know that it is NP-hard to compute $R_T(\sigma)$ for an arbitrary involution σ in I_{2n} even if T is a binary tree. By Proposition 2, we know that computing $R_T(\sigma)$ is $3/2-$approximable. The aim of this section is to find the average coloring number required for this approximation algorithm in the case of involutions with no fixed points and therefore to complete the work initiated in [20]. We compute the average load $\tilde{L}_T$ for any tree T and will obtain bounds on the average number $\tilde{R}_T$ of colors needed to color any involution-path set on T.

We can compute easily the average load of any arc i^+ of T, $1 \leq i \leq 2n-1$: $\tilde{L}_T(i) = |T(i)|(2n - |T(i)|)/(2n-1)$. Therefore, we obtain a lower bound for the average load $\tilde{L}_T$ and the following lower bound on the average number of colors needed to color any involution $\sigma \in I_{2n}$ on any tree T on $2n$ vertices.

Lemma 8 $\tilde{R}_T \geq 2n\tilde{v}_T(1-\tilde{v}_T)$.

By using a bijection between the involutions in I_{2n} and the set V_{2n} of special walks on $\mathbb{N}\times\mathbb{N}$ called *labeled Dyck walks* of length $2n$ [20, 27] that preserves the load as Biane's bijection does, and by Louchard's Theorem 5.3 [22], we get the following results that can be obtained by the same methods as in the previous sections for arbitrary permutations. Restating Louchard's Theorem 5.3, we get the following lemma :

Lemma 9 *Let P_{2n} be the path graph on $2n$ vertices and let σ^* be a random involution with no fixed points in I_{2n}. The load $L_{P_{2n}}(\sigma^*, \lfloor 2nv \rfloor)$ of any arc $\lfloor 2nv \rfloor$ of P_{2n} ($v \in [0,1]$) induced by σ^* has the following behavior*

$$L_{P_{2n}}(\sigma^*, \lfloor 2nv \rfloor) = 2nv(1-v) + X(v)\sqrt{n} + O(1),$$

where X is a Markovian process with mean 0 and covariance $C(s,t) = 2s^2(1-t)^2$, $s \leq t$.

Now we use Daniels and Skyrme's result [7] :

Theorem 10 *The average coloring number of involutions with no fixed points in I_{2n} to be routed on a linear network P_{2n} on $2n$ vertices is $\tilde{R}_{P_{2n}} = n/2 + \Theta(n^{1/3})$.*

Theorem 11 *The variance of the coloring number of involutions with no fixed points in I_{2n} to be routed on a linear network P_{2n} on $2n$ vertices is $n/4$.*

Note that the average complexity for involutions is the same as for permutations. We apply the relabeling argument used in Section 6.1 to generalize these results to any arbitrary tree.

Lemma 10 *Let T be a tree on $2n$ vertices and let σ^* be a random involution in I_{2n}. The load of any arc i^+ of T induced by σ^*, denoted by $L_T(\sigma^*, i^+)$, has the following behavior*

$$L_T(\sigma^*, i^+) = 2nv_T(i)(1-v_T(i)) + X(v_T(i))\sqrt{n} + O(1),$$

where X is a Markovian process with mean 0 and covariance $C(s,t) = 2s^2(1-t)^2$, $s \leq t$, and where $v_T(i) = |T(i)|/2n$.

Thanks to this lemma we compute the average load and the upper bound of the coloring number :

Theorem 12 *Let T be a tree on $2n$ vertices. The average load induced by all involutions with no fixed points $\sigma \in I_{2n}$ on T is $\tilde{L}_T = 2n\tilde{v}_T(1-\tilde{v}_T) + O(\sqrt{n})$.*

Theorem 13 *For all ϵ, there exists $n_0 = n_0(\epsilon)$ such that, for all $n \geq n_0$ and any tree T on $2n$ vertices, the average number of colors $\tilde{R}_T$ verifies, $\tilde{R}_T \leq \left(\frac{3}{2} + \epsilon\right) 2n\tilde{v}_T(1 - \tilde{v}_T)$.*

We simply apply Theorem 12 to get the average coloring number for involutions with no fixed points on generalized star tree networks obtaining exactly the same asymptotic behavior as for arbitrary permutations. Let k be a fixed integer greater than 2 and $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of $2n - 1$ into k parts.

Corollary 1 *The average coloring number of the involutions in I_{2n} to be routed on a generalized star network $GST(\lambda)$ is $\tilde{R}_{GST(\lambda)} = 2n\tilde{v}_{GST(\lambda)}(1 - \tilde{v}_{GST(\lambda)}) + O(\sqrt{n})$, with $\tilde{v}_{GST(\lambda)} = \min(n, \lambda_1)$.*

8 Open Problems

The complexity of routing circular permutations on both binary trees and on trees having exactly two vertices with degree greater than two is open. Another interesting open problem is to derive (whenever possible) a polynomial-time approximation algorithm for the general path coloring problem on directed symmetric trees with a better ratio than 5/3. Computing the average coloring number of permutations to be routed on arbitrary topology networks seems a very difficult problem. In fact, the results we obtain on the average coloring number of permutations to be routed on tree networks use a nice property of bounded ratio (less than 2) between, the coloring number and the load induced by a permutation-path set. As far as we known, in the case of arbitrary topology networks there is no result about such a ratio. Thus, studying the behavior of the ratio between the coloring number and the load induced by any permutation-path set on an arbitrary network can help us to derive a general approach to analyze the average behavior of the coloring number of routing permutations on networks.

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References

[1] Y. Aumann, Y. Rabani. *Improved bounds for all optical routing.* In Proc. of the 6th ACM-SIAM Symposium on Discrete Algorithms, pp 567-576, 1995.

[2] Ph. Biane. *Permutations suivant le type d'excédance et le nombre d'inversions et interprétation combinatoire d'une fraction continue de Heine.* Eur. J. Comb., 14(4):277-284, 1993.

[3] B. Bollobàs. *Graph Theory. An Introductory Course.* Graduate Texts in Mathematics no. 63, Springer-Verlag, 1979.

[4] I. Caragiannis, Ch. Kaklamanis, P. Persiano. *Wavelength Routing of Symmetric Communication Requests in Directed Fiber Trees.* In Proc. of the 5th International Colloquium on Structural Information and Communication Complexity (SIROCCO). Carleton Scientific, 1998.

[5] N. K. Cheung, K. Nosu, and G. Winzer, editors. *Special Issue on Dense Wavelength Division Multiplexing Techniques for High Capacity and Multiple Access Communications Systems.* IEEE Journal on Selected Areas in Communications, 8(6), 1990.

[6] H. E. Daniels, *The maximum of a gaussian process whose mean path has a maximum, with an application to the strength of bundles of fibres.* Adv. Appl. Probab., 21:315-333, 1989.

[7] H. E. Daniels, T. H. R. Skyrme. *The maximum of a random walk whose mean path has a maximum.* Adv. Appl. Probab., 17:85-99, 1985.

[8] T. Erlebach, K. Jansen. *Call scheduling in trees, rings and meshes.* In Proc. of the 30th Hawaii International Conference on System Sciences (HICSS-30), vol. 1, pp 221-222. IEEE CS Press, 1997.

[9] T. Erlebach, K. Jansen, Ch. Kaklamanis, M. Mihail, P. Persiano. *Optimal wavelength routing on directed fiber trees.* Theoret. Comput. Sci., 221(1-2):119-137, 1999.

[10] Ph. Flajolet. *Combinatorial aspects of continued fractions.* Discrete Math., 32:125-161, 1980.

[11] H. Fleischner. *Eulerian graphs and related topics.* Annals of Discrete Mathematics, no. 45, vol. 2, Elsevier Science Publisher B.V., 1991.

[12] J. Françon, X. Viennot. *Permutations selon leurs pics, creux, doubles montées et doubles descentes, nombres d'Euler et nombres de Genocchi.* Discrete Math., 28:21-35, 1979.

[13] D. Gardy, G. Louchard. *Dynamic analysis of some relational databases parameters.* Theoret. Comput. Sci., 144:125-159, 1995.

[14] M. R. Garey, D. S. Johnson, G. L. Miller, Ch. Papadimitriou. *The complexity of coloring circular arcs and chords.* SIAM J. Alg. Disc. Meth., 1(2):216-227,1980.

[15] L. Gargano, P. Hell, S. Perennes. *Coloring all directed paths in a symmetric tree with applications to WDM routing.* In Proc. of 24th International Colloquium on Automata, Languages, and Programming, LNCS 1256, pp 505-515, 1997.

[16] M. C. Golumbic. *Algorithmic graph theory and perfect graphs.* Academic Press, New York, 1980.

[17] Q.-P. Gu, H. Tamaki. *Routing a permutation in the Hypercube by two sets of edge-disjoint paths.* J. of Parallel and Distributed Comput. 44(2):147-152, August 1997.

[18] U. I. Gupta, D. T. Lee, J. Y.-T. Leung. *Efficient algorithms for interval graphs and circular-arc graphs.* Networks, 12:459-467, 1982.

[19] S. R. Kumar, R. Panigrahy, A. Russel, R. Sundaram. *A note on optical routing on trees.* Inf. Process. Lett., 62:295-300, 1997.

[20] J.C. Lagarias, A.M. Odlyzko, D.B. Zagier. *On the Capacity of Disjointly Shared Networks.* Computer Networks and ISDN Syst. 10(5):275-285,1985.

[21] F. T. Leighton, S. Rao. *An approximate max-flow min-cut theorem for uniform multicommodity flow problems with applications to approximation algorithms.* In Proc. of 29th IEEE FOCS, pp 422-431, 1988.

[22] G. Louchard. *Random walks, Gaussian processes and list structures.* Theoret. Comput. Sci., 53:99-124, 1987.

[23] D. W. Matula, F. Shahrokhi. *The Maximum Concurrent Flow Problem and Sparsest Cuts.* Technical Report, Southern Methodist University, March 1986.

[24] M. Paterson, H. Schröder, O. Sýkora, I. Vrťo. *On Permutation Communications in All-Optical Rings.* In Proc. of the 5th International Colloquium on Structural Information and Communication Complexity (SIROCCO). Carleton Scientific, 1998.

[25] P. Raghavan, E. Upfal. *Efficient routing in all optical networks.* In Proc. of the 26th ACM Symposium on Theory of Computing, pp 133-143, 1994.

[26] A. Tucker. *Coloring a family of circular arcs.* SIAM J. Appl. Maths., 29(3):493-502, 1975.

[27] X. Viennot. *A combinatorial theory for general orthogonal polynomials with extensions and applications.* Lect. Notes Math., 1171:139-157, 1985.

[28] G. Wilfong, P. Winkler. *Ring routing and wavelength translation.* In Proc. of the 9th Annual ACM-SIAM Symposium on Discrete Algorithms, pp 333-341, 1998.